On Nagata's Conjecture*

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This paper gives an improved lower bound on the degrees d such that for general points $p_1, \ldots, p_n \in \mathbf{P}^2$ and $m > 0$ there is a plane curve of degree d vanishing at each p_i with multiplicity at least m.

Key Words: multiple points, plane curves, fat points, Nagata's conjecture, regularity.

1. INTRODUCTION

In this paper we work over an arbitrary algebraically closed field. For positive integers m and n, define $\delta(m, n)$ to be the least integer d such that for general points $p_1, \ldots, p_n \in \mathbf{P}^2$ (i.e., for an open dense set, depending on m and n, of n points of $\mathbf{P}^2$) there is a curve of degree d vanishing at each point p_i with multiplicity at least m. For $n \geq 10$, Nagata [11] conjectures that $\delta(m, n) > m\sqrt{n}$, and proves this when $n > 9$ is a square. (For $n \leq 9$, applying methods of [12] it can be shown that $\delta(m, n) = \lceil c_n m \rceil$, where $c_n = 1, 1, 1.5, 2, 2, 12/5, 21/8, 48/17$ and 3 for $n = 1, \ldots, 9$, resp. Recall for any real number c that $\lfloor c \rfloor$ is the greatest integer less than or equal to c and $\lceil c \rceil$ is the least integer greater than or equal to c; in particular, $\lfloor c \rfloor \leq c \leq \lceil c \rceil$.)

Clearly, if $n' \leq n$, then $\delta(m, n') \leq \delta(m, n)$, so we see from Nagata's result above that $\delta(m, n) \geq \delta(m, \lfloor \sqrt{n} \rfloor^2) > m\lfloor \sqrt{n} \rfloor$ for $n \geq 16$. In fact, it is not hard to show directly for all $n \geq 1$ the slightly weaker inequality $\delta(m, n) \geq m\lfloor \sqrt{n} \rfloor$; similar reasoning shows $\delta(m, n) \geq mn/\lceil \sqrt{n} \rceil$ as well (see

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Lemma 2.1). In certain ranges of n, however, Roé [13] has recently given a better bound: for $n \geq 3$ he shows that $\delta(m,n) \geq mr(n)$, where Roé's constant $r(n)$ is defined as $r(n) = (n-1)\Pi_{i=2}^{n-1}(1-i/(n-1+i^2))$. Roé applies a procedure, called unloading, to an astute sequence of specializations, to derive an algorithm for computing a value $R(m,n)$ depending on m and n. It turns out on general principles that $\delta(m,n) \geq R(m,n)$; the bound $\delta(m,n) \geq mr(n)$ is obtained by showing that $R(m,n) \geq mr(n)$.

Although it seems hard actually to prove that $R(m,n) > m\sqrt{n}$ for $m < \sqrt{n}$, examples suggest that this is at least typically true. Indeed, a direct check shows for $2 \leq m \leq 100$ that $R(m,m^2)$ is, plus or minus at most 1, equal to $m^2 + m/10$. (In a personal communication, Prof. Roé has told me that in fact $R(m,m^2) \geq m^2 + \lfloor m/10 \rfloor$ for m up to 200.) These and other data indicate that Roé's result $\delta(m,n) \geq R(m,n)$ is the best general bound currently known when m is not too large compared to n. For comparison, [3] proves Nagata's conjecture for values of m up to about $\sqrt{n/2}$, and, in characteristic 0, [4] determines $\delta(m,n)$ for any m when n is a power of 4, while [1] and [2] do so for $m \leq 12$ and $n \geq 10$. These exact values agree with conjectures (see [5], [10], [6], [1], [2], [9]) which imply for $n \geq 10$ that $\delta(m,n)$ should be the least positive integer d such that $d^2 + 3d + 2 - nm^2 - mn > 0$. When n is an even square and $m \geq (\sqrt{n}-2)/4$, this d is precisely $m\sqrt{n} + (\sqrt{n}-2)/2$ (see [9]), which unfortunately tends to be somewhat larger than $R(m,n)$. (As an aside, we mention that the current paper resulted from this author's wondering whether Roé's algorithm might in some cases be used to justify $\delta(m,n) = m\sqrt{n} + (\sqrt{n}-2)/2$ when n is an even square and $m \geq (\sqrt{n}-2)/4$, in which case the results of [9] would determine a minimal free resolution of the m-th symbolic power $I(m,n)$ of the ideal generated by all forms vanishing at n general points of $\mathbf{P}^2$. As discussed in [9], to write down the resolution of $I(m,n)$ it is enough to know two things: its Hilbert function $h_{m,n}(t) = \dim I(m,n)_t$, which gives the dimension of the homogeneous component $I(m,n)_t$ of $I(m,n)$ in each degree t, and the number ν_t of generators of $I(m,n)$ in each degree t in any minimal set of homogeneous generators for $I(m,n)$. Bounds on $\delta(m,n)$ give partial information on the resolution, since clearly $h_{m,n}(t) = 0 = \nu_t$ for $t < \delta(m,n)$, and $\nu_{\delta(m,n)} = h_{m,n}(\delta(m,n))$. The case that n is an even square bigger than 9 is especially interesting since then, for m sufficiently large, $\delta(m,n)$ is expected to be $m\sqrt{n} + (\sqrt{n}-2)/2$, and [9] determines $h_{m,n}(t)$ for all $t \geq m\sqrt{n} + (\sqrt{n}-2)/2$ and ν_t for all $t > m\sqrt{n} + (\sqrt{n}-2)/2$. Unfortunately, except in special cases, when $n > 9$ is a square Nagata's result showing $\delta(m,n) \geq m\sqrt{n} + 1$ remains the best bound known. Thus the results of this paper are of interest mainly when n is not a square.)

In Section 2 we set up the context in which we will obtain our results, and we recall some previously known bounds on $\delta(m,n)$. In Section 3, using a single specialization inspired by Roé's, we obtain our main result, Theorem 3.2, which shows that

$$\delta(m,n) \geq \lceil m\lambda_n \rceil,$$

where λ_n denotes $n\lfloor\sqrt{n}\rfloor/\lceil\sqrt{n}\lfloor\sqrt{n}\rfloor\rceil$. In Section 4 we show that this is an improvement on the bounds previously known. In particular, we verify that:

- $\lambda_n \geq \lfloor\sqrt{n}\rfloor$ for all $n \geq 1$, with equality if and only if n or $n-1$ is a square;
- $\lambda_n \geq n/\lceil\sqrt{n}\rceil$ for all $n \geq 1$, with equality if and only if n, $n+1$ or $n+2$ is a square;
- $\lambda_n > r(n)$ for all $n \geq 3$; and
- for each $n \geq 3$, that $\lim_{m\to\infty}(m\lambda_n - R(m,n)) = \infty$.

In Section 5 we show for certain values of n with m not too large, that our bound implies Nagata's conjecture.

2. BACKGROUND

We refer the reader to [8] for justification and amplification of the material in this section. Given essentially distinct points $p_1,\ldots,p_n \in \mathbf{P}^2$ (meaning for $i = 0,\ldots,n-1$ that $p_{i+1} \in X_i$ where $X_0 = \mathbf{P}^2$ and $\pi_{i+1} : X_{i+1} \to X_i$ is the blow up of p_{i+1}), we will denote X_n simply by X, with the morphism $\pi : X \to X_0$ being the composition $\pi_n \circ \ldots \circ \pi_1$ of the blow ups. The inverse image of p_i with respect to π_i is a divisor on X_i; the class of the total transform to X of this divisor will be denoted e_i. The class of the total transform to X of a line in $\mathbf{P}^2 = X_0$ will be denoted e_0. The divisor class group on X is then freely generated by the classes e_i, $i = 0,\ldots,n$, with the intersection form being defined by $e_i \cdot e_j = 0$ for $i \neq j$, $e_0^2 = 1$ and $e_i^2 = -1$ for $i > 0$.

Define $\delta_X(m,n)$ to be the least t such that $h^0(X, \mathcal{O}_X(te_0 - m(e_1+\cdots+e_n))) > 0$. Then $\delta(m,n)$ is the maximum value of $\delta_X(m,n)$ over all essentially distinct sets of n points of $\mathbf{P}^2$. (By semicontinuity, it follows that $\delta(m,n) = \delta_X(m,n)$ for a general set of distinct points $p_1,\ldots,p_n$.) To give a bound $\delta(m,n) \geq d$, it clearly suffices to find a d and an X for which we can check $\delta_X(m,n) \geq d$ (i.e., for which $h^0(X, \mathcal{O}_X((d-1)e_0 - m(e_1+\cdots+e_n))) = 0$). This follows, for example, if X has a numerically effective (also called *nef*) divisor C such that $C \cdot ((d-1)e_0 - m(e_1+\cdots+e_n)) < 0$. The following lemma, which is well-known, is, as we show, easy to prove this way. (The slightly stronger result $\delta(m,n) > m\lfloor\sqrt{n}\rfloor$ which follows from [11] as mentioned above, requires a related but somewhat more involved argument.)

Lemma 2.1. *Let m and n be positive integers. Then we have:*

(a) $\delta(m,n) \geq m\lfloor\sqrt{n}\rfloor$, *and*
(b) $\delta(m,n) \geq mn/\lceil\sqrt{n}\rceil$.

Proof. To prove (a), choose distinct points $p_1,\dots,p_{r^2}$ of a smooth irreducible plane r-ic C', with $r = \lfloor\sqrt{n}\rfloor$. Let X be the surface obtained by blowing up $\mathbf{P}^2$ at $p_1,\dots,p_{r^2}$ and let C be the class of the proper transform to X of C'. Then C (being reduced and irreducible with $C^2 \geq 0$) is numerically effective; i.e., by definition $C \cdot F \geq 0$ for every class F on X with $h^0(X, \mathcal{O}_X(F)) > 0$ (we will refer to such a class F as an *effective class*). In particular, $\delta(m,n)e_0 - m(e_1 + \cdots + e_{r^2})$ is effective since $\delta(m,n) \geq \delta(m,r^2)$ and since $\delta(m,r^2)e_0 - m(e_1 + \cdots + e_{r^2})$ is effective, so we have $\delta(m,n)r \geq mr^2$, and hence $\delta(m,n) \geq mr = m\lfloor\sqrt{n}\rfloor$.

To prove (b), choose distinct points $p_1,\dots,p_n$ of a smooth irreducible plane r-ic C', where this time $r = \lceil\sqrt{n}\rceil$ and X is the surface obtained by blowing up $\mathbf{P}^2$ at $p_1,\dots,p_n$ and C is the class of the proper transform to X of C'. Then reasoning as above gives $\delta(m,n)r \geq mn$, and hence the result. ∎

3. THE MAIN RESULT

In this section, we use a special arrangement of essentially distinct points, similar to what is used in [13], to which we will apply an argument analogous to that used in the proof of Lemma 2.1.

Proposition 3.1. *Let d, n and r be positive integers such that $(r/d)^2 \geq n$ and $r \leq n$. Then $\delta(m,n) \geq mnd/r$.*

Proof. Let C_1 be a smooth plane curve of degree d. Choose any point p such that $p_1 \in C_1 \subset X_0 = \mathbf{P}^2$. Let X_1 be the blow up of X_0 at p_1, and let C_2 be the proper transform of C_1. Then choose p_2 to be the point of C_2 infinitely near to p_1. Continue in this way, iteratively obtaining essentially distinct points p_i, $i = 1,\dots,r$, where, for $1 < i \leq r$, p_i is the point of C_i infinitely near to p_{i-1} with respect to the blowing up $\pi_{i-1} : X_{i-1} \to X_{i-2}$ of p_{i-1}, with C_i being the proper transform of C_{i-1} with respect to π_{i-1}. (Thus C_1 and p_1 determine p_i for $1 < i \leq r$.)

If $n > r$, choose additional points $p_{r+1},\dots,p_n$ so that again each point p_i is infinitely near to p_{i-1} for $i \geq r+1$ but so that p_{r+1} is not on the proper transform of C_r and none of p_i, $i \geq r+1$ is on the proper transform to X_{i-1} of the exceptional locus of the blow up morphism $X_{i-2} \to X_{i-3}$ (i.e., p_i is chosen so that $e_{i-1} - e_i$ but not $e_{i-2} - e_{i-1} - e_i$ is effective). As usual, we denote X_n by X; C will denote the class of the proper transform of C_1 to X.

Then C is the class of a smooth, irreducible curve, as is each of $e_1 - e_2, \dots, e_{n-1} - e_n$ and e_n. By hypothesis, $d^2 n \le r^2$ and $r \le n$ and hence $d^2 \le r$; using $d^2 \le r$, it is not hard to verify that $rde_0 - d^2(e_1 + \cdots + e_r) - (r^2 - rd^2)e_{r+1}$ is the sum of rC and various nonnegative multiples of $e_i - e_{i+1}$ for $1 \le i \le r$ (here we have assumed that $r < n$; we leave it to the reader to consider the case that $r = n$). But $d^2 n \le r^2$ implies $r^2 - rd^2 \ge (n-r)d^2$, hence the class $D = rde_0 - d^2(e_1 + \cdots + e_n)$ is the sum of $rde_0 - d^2(e_1 + \cdots + e_r) - (r^2 - rd^2)e_{r+1}$ and various nonnegative multiples of e_n and of $e_i - e_{i+1}$ for $r+1 \le i \le n-1$. Thus D is effective. But $D \cdot C = 0$, $D \cdot (e_i - e_{i+1}) = 0$ for $i > 0$ and $D \cdot e_n \ge 0$, so D, being a sum of effective classes which it meets nonnegatively, is nef. Therefore, $\delta(m,n)rd - d^2 mn = (\delta(m,n)e_0 - m(e_1 + \cdots + e_n)) \cdot D \ge 0$; i.e., $\delta(m,n) \ge mnd/r$, as claimed. ∎

As a corollary we derive:

Theorem 3.2. *Let n and m be positive integers; then $\delta(m,n) \ge \lceil m\lambda_n \rceil$.*

Proof. Apply Proposition 3.1 with $d = \lfloor \sqrt{n} \rfloor$ and $r = \lceil \sqrt{n} d \rceil$. We merely need to check that $(r/d)^2 \ge n$ and $r \le n$. Clearly, $(r/d)^2 \ge (\sqrt{n} d/d)^2 = n$. For the other inequality, we have $\sqrt{n}\lfloor \sqrt{n} \rfloor \le \sqrt{n}\sqrt{n} = n$, hence, since n is an integer, $r = \lceil \sqrt{n} \lfloor \sqrt{n} \rfloor \rceil \le n$, as required. ∎

It does not seem easy to know a priori what choice of r and d is best. We can always write any given positive integer n as $n = s^2 + t$, where $s = \lfloor \sqrt{n} \rfloor$ and $t = n - \lfloor \sqrt{n} \rfloor^2$, hence t is between 0 and $2s$. The choice $d = \lfloor \sqrt{n} \rfloor$ and $r = \lceil \sqrt{n} \lfloor \sqrt{n} \rfloor \rceil$, which gives λ_n, amounts to taking $d = s$ and $r = s^2 + \lceil t/2 \rceil$ (see the proof of Proposition 4.1), but this is not always the best choice. For example, if $1 < s \le t < 2s - 1$ with t odd, it is easy to check that $d = s - 1$ and $r = s(s-1) + (t-1)/2$ satisfy the hypotheses of Proposition 3.1 but give a slightly better bound on $\delta(m,n)$ than does λ_n. However, in terms of both simplicity and general applicability, the author knows of no better choice than $d = \lfloor \sqrt{n} \rfloor$ and $r = \lceil \sqrt{n} \lfloor \sqrt{n} \rfloor \rceil$.

4. COMPARISONS

We begin by comparing λ_n with $\lfloor \sqrt{n} \rfloor$ and $n / \lceil \sqrt{n} \rceil$. We will use repeatedly the easy fact that any integer $n \ge 0$ can be (uniquely) written in the form $n = s^2 + t$, where s is a nonnegative integer and $0 \le t \le 2s$ (indeed, s is just $\lfloor \sqrt{n} \rfloor$).

Proposition 4.1. *Let $n = s^2 + t$ where $s > 0$ and $0 \le t \le 2s$ are integers; then:*

(a) $\lambda_n \geq \lfloor\sqrt{n}\rfloor$, *with equality if and only if* $t = 0$ *or* $t = 1$, *and*

(b) $\lambda_n \geq n/\lceil\sqrt{n}\rceil$, *with equality if and only if* $t = 0$, $t = 2s-1$ *or* $t = 2s$.

Proof. First, one sees that $\lfloor\sqrt{n}\rfloor = s$. With slightly more trouble, one also checks that $\lceil\sqrt{n}\lfloor\sqrt{n}\rfloor\rceil = s^2 + \lceil t/2\rceil$. It is now easy to see that $(s^2 + t)s/(s^2 + \lceil t/2\rceil) \geq s$, with equality if and only if t is 0 or 1; this is (a). Next, if $t = 0$, (b) is true, while for $t > 0$, we have $(s^2 + t)s/(s^2 + \lceil t/2\rceil) \geq (s^2 + t)/(s+1)$, with equality if and only if t is $2s$ or $2s-1$, which is the rest of (b). ▮

We next want to compare $m\lambda_n$ with Roé's bounds $mr(n)$ and $R(m,n)$. In order to deal with $R(m,n)$ it will be helpful to describe Roé's algorithm for computing it.

We first develop some notation and terminology. Let $\mathbf{w} = (m_1, \ldots, m_n)$ be a vector; then $p(\mathbf{w})$ will denote the vector obtained from $\mathbf{w}$ by putting the entries m_j with $j > 1$ into nonincreasing order. We will use $\mathbf{v}_i$ to denote the vector $(1, -1, \ldots, -1, 0, \ldots, 0)$, where there are i entries of -1. Replacing every negative entry of a vector by 0 we will call *rectification*. We will define $q_i(\mathbf{w})$ to be $\mathbf{w}$, if, with respect to the usual dot product, $\mathbf{w} \cdot \mathbf{v}_i \geq 0$; otherwise $q_i(\mathbf{w})$ will be the rectification of $\mathbf{w} + \mathbf{v}_i$.

Now let $n \geq 3$ be an integer; for each integer i with $2 \leq i \leq n-1$ we describe a routine $\mathcal{O}_i$. Given a vector $\mathbf{w} = (m_1, \ldots, m_n)$ of nonnegative integers, let g_i denote the composition pq_ip, so $g_i(\mathbf{w}) = pq_ip(\mathbf{w})$, and consider the sequence $g_i(\mathbf{w}), g_ig_i(\mathbf{w}), \ldots$. It is easy to see that eventually the sequence stabilizes at some vector which we will denote by $\mathcal{O}_i(\mathbf{w})$.

Roé's algorithm, then, is to apply the routines $\mathcal{O}_2, \ldots, \mathcal{O}_{n-1}$ consecutively to an initial input vector $\mathbf{w} = (m, \ldots, m)$; the value $R(m,n)$ is the first entry of the vector $\mathcal{O}_{n-1} \cdots \mathcal{O}_2(\mathbf{w})$.

Since $\mathbf{w} = (m, \ldots, m)$ is of particular interest, in this case we will denote the first component of $\mathcal{O}_i \cdots \mathcal{O}_2(\mathbf{w})$ by $R_i(m,n)$ and set $R_1(m,n) = m$; thus $R_{n-1}(m,n) = R(m,n)$. The sum of the second through nth components of $\mathcal{O}_i \cdots \mathcal{O}_2(\mathbf{w})$ will be denoted by $S_i(m,n)$ and we set $S_1(m,n) = (n-1)m$. Suppressing m and n when no confusion will result, we may write R_i or S_i instead of $R_i(m,n)$ or $S_i(m,n)$.

Another description of the algorithm will be helpful. Given integers $n > 1$ and $1 \leq c \leq n-1$, we will use $v(a,b,c,n)$ to denote the vector $(a, b, \ldots, b, b-1, \ldots, b-1)$, where there are c of the b entries and n entries altogether. For example, if $\mathbf{w} = (m, \ldots, m)$, then $\mathbf{w} = v(m, m, n-1, n)$.

Now let $\mathbf{v} = v(m_1, b, c, n)$ with $b \geq 1$ and assume that $\mathbf{v} \cdot \mathbf{v}_{i-1} \geq 0$. Then $g_i(\mathbf{v}) = \mathbf{v}$ if $\mathbf{v} \cdot \mathbf{v}_i \geq 0$ but if $\mathbf{v} \cdot \mathbf{v}_i < 0$ then $g_i(\mathbf{v})$ is $v(m_1 + 1, b', c', n)$, where $b' = b$ if $i < c$, in which case $c' = c - i$, and $b' = b - 1$ if $i \geq c$, in which case $c' = n - i + c$. (Because of possible rectification, the case that

$b = 1$ is a bit tricky, but since $\mathbf{v} \cdot \mathbf{v}_{i-1} \geq 0$, if $b = 1$ and $\mathbf{v} \cdot \mathbf{v}_i < 0$, then $i \leq c$, so no negative entries are ever involved.)

In Roé's algorithm, we continue to apply g_i until the dot product with $\mathbf{v}_i$ becomes nonnegative. If starting with $\mathbf{v}$, t is the least number of such applications required for the dot product with $\mathbf{v}_i$ to become nonnegative, then (denoting by S the sum $(n-1)(b-1)+c$ of all components of $\mathbf{v}$ but the first) the result of applying g_i for $j \leq t$ times is $v(m_1+j, \lceil (S-ji)/(n-1) \rceil, \rho_j, n)$, where $\rho_j = (S-ji) - (n-1)\lfloor (S-ji)/(n-1) \rfloor$ is the remainder when $S - ji$ is divided by $n-1$. Looking at $\mathbf{v}_i \cdot v(m_1+j, \lceil (S-ji)/(n-1) \rceil, \rho_j, n)$ we see that t is the least integer j such that $i\lfloor (S-ji)/(n-1) \rfloor + \min(\rho_j, i) \leq m_1 + j \leq i\lfloor (S-(j-1)i)/(n-1) \rfloor + \min(\rho_{j-1}, i)$.

In particular, applying the above remarks to

$$v(R_{i-1}, b, c, n) = \mathcal{O}_{i-1} \cdots \mathcal{O}_2(\mathbf{w})$$

and $v(R_i, b', c', n) = \mathcal{O}_i \cdots \mathcal{O}_2(\mathbf{w})$, where $\mathbf{w} = (m, \dots, m)$, we have the following formulas:

(F1) $R_i = R_{i-1} + t$ where t is the least j such that $i\lfloor (S_{i-1} - ji)/(n-1) \rfloor + \min(\rho_j, i) \leq R_{i-1} + j \leq i\lfloor (S_{i-1} - (j-1)i)/(n-1) \rfloor + \min(\rho_{j-1}, i)$;

(F2) $S_{i-1} = (b-1)(n-1) + c$; and

(F3) $S_i = S_{i-1} - i(R_i - R_{i-1})$ or equivalently $R_i + S_i/i = R_{i-1} + S_{i-1}/i$ (which are the same as $i(R_i - R_{i-1}) = S_{i-1} - S_i$, which holds since R increases by 1 for each decrease in S by i).

We note that the value $mr(n)$ can be obtained by a similar but "averaged" procedure, which requires working over the rationals. In place of $\mathbf{v}_i$ we have $\overline{\mathbf{v}}_i = (1, -i/(n-1), \dots, -i/(n-1))$, and in place of q_i we have $\overline{q}_i$, where $\overline{q}_i(\mathbf{w})$ is $\mathbf{w}$ if $\mathbf{w} \cdot \overline{\mathbf{v}}_i \geq 0$; otherwise $\overline{q}_i(\mathbf{w})$ is the rectification of $\mathbf{w} + t\overline{\mathbf{v}}_i$, where t is chosen so that $(\mathbf{w} + t\overline{\mathbf{v}}_i) \cdot \overline{\mathbf{v}}_i = 0$. We define $\overline{g}_i$ to be $p\overline{q}_i p$ (we use p simply for analogy; because of the averaging, nothing important would be affected if we did not use it), and we take $\overline{\mathcal{O}}_i(\mathbf{w})$ to be the vector at which the sequence $\overline{g}_i(\mathbf{w}), \overline{g}_i\overline{g}_i(\mathbf{w}), \dots$ stabilizes. (Note that $\overline{\mathcal{O}}_i(\mathbf{w}) = \overline{g}_i(\mathbf{w})$ if neither $\mathbf{w}$ nor $\overline{q}_i(\mathbf{w})$ has a negative entry.)

Now let m be a positive integer, let $r_1(n) = 1$, let $mr_i(n)$ be the first entry of $\mathbf{w}_i = \overline{\mathcal{O}}_i \cdots \overline{\mathcal{O}}_2(\mathbf{w}_1)$, where $\mathbf{w}_1$ is the n-vector $(m, \dots, m)$, let $s_1(n) = n - 1$ and let $ms_i(n)$ be the sum of all of the entries but the first of $\mathbf{w}_i$. It is not hard by induction to check that

(f0) $\mathbf{w}_i \cdot \mathbf{v}_i = 0$ for $2 \leq i \leq n-1$ and hence $\mathbf{w}_i \cdot \mathbf{v}_{i+1} < 0$ for $1 \leq i < n-1$, and that

(f1) $r_i(n) = r_{i-1}(n)(i/(i-1))(1 - i/(n-1+i^2))$, and

(f2) $s_i(n) = ((n-1)/i)r_i(n)$, from which it follows that

(f3) $r_i(n) + s_i(n)/i = r_{i-1}(n) + s_{i-1}(n)/i$ and that (f4) $r_i(n) = (i^2/(n-1+i^2))(r_{i-1} + s_{i-1}/i)$.

By (f1), of course, we have $r_i(n) = \Pi_{j=2}^{i}(j(1-j/(n-1+j^2))/(j-1))$ and hence $r(n) = r_{n-1}(n)$.

Proposition 4.2. *Let $n \geq 3$ be an integer; then:*

(a) $r(n) \leq \sqrt{n-1} - \pi/8 + 1/\sqrt{n-1}$, and
(b) $R(m,n) \leq mr(n) + 2(n-1)$.

Proof. On behalf of easier reading, we will in this proof use k to denote $n-1$. By direct check, (a) holds for $2 \leq k \leq 3$. So assume $k \geq 4$. In any case, $r(n) = k\Pi_{i=2}^{k}(1 - i/(k+i^2))$, and [13] shows that $(r(n))^2 = k\Pi_{i=1}^{k-1}(1-(i/(k+i^2))^2)$. But $\log(1-x) < -x$ holds for $0 < x < 1$, so we have $\sum_{i=1}^{k-1}\log(1-(i/(i^2+k))^2) \leq -\sum_{i=1}^{k-1}(i/(i^2+k))^2 = -\sum_{i\geq 1}(i/(i^2+k))^2 + \sum_{i\geq k}(i/(i^2+k))^2$. However, $\sum_{i\geq k}(i/(i^2+k))^2 \leq \int_{i\geq k-1}(x/(x^2+k))^2 dx \leq \int_{i\geq k-1} x^{-2}dx = 1/(k-1)$, and from [13] we see that $\sum_{i\geq 1}(i/(i^2+k))^2 = \pi(-\pi + (\sinh(2\sqrt{k}\pi))/(2\sqrt{k}))/(4\sinh^2(\sqrt{k}\pi)) = -\pi^2/(4\sinh^2(\sqrt{k}\pi)) + (\pi/(8\sqrt{k}))(\sinh(2\sqrt{k}\pi))/\sinh^2(\sqrt{k}\pi)$. But

$$(\pi/(8\sqrt{k}))(\sinh(2\sqrt{k}\pi))/\sinh^2(\sqrt{k}\pi)$$

equals $(\pi/(4\sqrt{k}))(1+\exp(-2\sqrt{k}\pi))/(1-\exp(-2\sqrt{k}\pi))$ which is at least as big as $(\pi/(4\sqrt{k}))(1+\exp(-2\sqrt{k}\pi))^2$, so $-\sum_{i=1}^{k-1}(i/(i^2+k))^2 \leq 1/(k-1) + \pi^2/(4\sinh^2(\sqrt{k}\pi)) - (\pi/(4\sqrt{k}))(1+\exp(-2\sqrt{k}\pi))^2 \leq 1/(k-1) + \pi^2/(4\sinh^2(\sqrt{k}\pi)) - (\pi/(4\sqrt{k})) \leq 1.1/(k-1) - (\pi/(4\sqrt{k}))$, where

$$\pi^2/(4\sinh^2(\sqrt{k}\pi)) < 0.1/(k-1)$$

follows from $(10\pi^2/4)(k-1) < \sinh^2(\sqrt{k}\pi)$, which itself is easy to check (look at a graph first). Since $1.1/(k-1) - \pi/(4\sqrt{k})$ is negative for $k \geq 4$, the Taylor series for $\exp(1.1/(k-1) - \pi/(4\sqrt{k}))$ is alternating so $(r(n))^2 \leq k\exp(1.1/(k-1) - \pi/(4\sqrt{k})) \leq k(1+1.1/(k-1) - \pi/(4\sqrt{k}) + (1/2)(1.1/(k-1) - \pi/(4\sqrt{k}))^2)$, but $1.1/(k-1) + (1/2)(1.1/(k-1) - \pi/(4\sqrt{k}))^2 < 1.1/(k-1) + (1/2)(\pi/(4\sqrt{k}))^2 < 2/k$ so $(r(n))^2 \leq k(1+2/k - \pi/(4\sqrt{k}))$, hence $r(n) \leq \sqrt{k(1+2/k-\pi/(4\sqrt{k}))} \leq \sqrt{k}(1+1/k-\pi/(8\sqrt{k})) = \sqrt{n-1} + 1/\sqrt{n-1} - \pi/8$.

Now consider (b). We begin by showing $R_i \leq (i^2R_{i-1} + iS_{i-1} + i^2 + ik)/(i^2+k)$. Let $t = R_i - R_{i-1}$. By (F1), we have $R_{i-1} + t \leq i\lfloor (S_{i-1} - (t-1)i)/k \rfloor + \min(\rho_{t-1}, i)$ and hence $R_{i-1} + t \leq i\lfloor (S_{i-1} - (t-1)i)/k \rfloor + i$. But $i\lfloor (S_{i-1} - (t-1)i)/k \rfloor + i \leq i(S_{i-1} - ti)/k + (i^2 + ki)/k$, so solving for t

gives $t \le (iS_{i-1} - kR_{i-1} + i^2 + ki)/(k + i^2)$ and therefore $R_i = R_{i-1} + t \le (iS_{i-1} + i^2R_{i-1} + i^2 + ki)/(k + i^2)$, as claimed.

Now (given m and n, and suppressing the n notationally) it will be sufficient to prove by induction for each i that $R_i \le mr_i + 2k$ and $R_i + S_i/i \le mr_i + s_i/i + 2k$. Note that $R_1 = m \le m + 2k = mr_1 + 2k$, and $R_1 + S_1/1 = nm \le nm + 2k = mr_1 + ms_1/1 + 2k$. So assume that $R_{i-1} \le mr_{i-1} + 2k$ and $R_{i-1} + S_{i-1}/(i-1) \le mr_{i-1} + ms_{i-1}/(i-1) + 2k$ hold for some $i \ge 2$.

Since $R_{i-1} \le mr_{i-1} + 2k$ and $R_{i-1} + S_{i-1}/(i-1) \le mr_{i-1} + ms_{i-1}/(i-1) + 2k$, then $R_{i-1} + S_{i-1}/i \le mr_{i-1} + ms_{i-1}/i + 2k$ must also hold, and using (F3) and (f3) we therefore have $R_i + S_i/i = R_{i-1} + S_{i-1}/i \le mr_{i-1} + ms_{i-1}/i + 2k = mr_i + ms_i/i + 2k$, as required.

Since $R_{i-1} + S_{i-1}/i \le mr_{i-1} + ms_{i-1}/i + 2k$ and since $R_i \le (iS_{i-1} + i^2R_{i-1}+i^2+ki)/(k+i^2) = (i^2/(k+i^2))(R_{i-1}+S_{i-1}/i)+(i^2+ik)/(i^2+k)$, the latter is at most $(i^2/(k+i^2))(mr_{i-1}+ms_{i-1}/i)+(2ki^2+i^2+ik)/(i^2+k)$, so by (f4), this latter simplifies to $mr_i+(2ki^2+i^2+ik)/(i^2+k)$ which (taking k to be i) is at most $mr_i+(2k^3+2k^2)/(k^2+k) = mr_i+2k$, as we needed to show. ∎

We now compare our bound with those of Roé.

Proposition 4.3. *Let $n \ge 3$ be an integer; then:*

(a) $\lambda_n > r(n)$, and

(b) $lim_{m\to\infty} m\lambda_n - R(m,n) = \infty$. In particular, $\lceil m\lambda_n \rceil > R(m,n)$ for all sufficiently large m.

Proof. Let $s = \lfloor\sqrt{n}\rfloor$ and write $n = s^2 + t$; thus $0 \le t \le 2s$ and $\lambda_n = (s^2+t)s/\lceil s\sqrt{n}\rceil \ge (s^2+t)s/\lceil s(s+t/(2s))\rceil = (s^2+t)s/\lceil s^2+t/2\rceil \ge (s^2+t)s/(s^2+(t+1)/2)$.

For part (a), one first checks case by case that $r(n) < \lambda_n$ for $3 \le n \le 48$, so we are reduced to the case that $n \ge 49$; i.e., $s \ge 7$. First assume $t = 0$; then in fact $\lambda_n = \sqrt{n}$. But since $n \ge 8$ we see $-\pi/8 + 1/\sqrt{n-1} < 0$ so, by Proposition 4.2(a), $r(n) \le \sqrt{n-1} - \pi/8 + 1/\sqrt{n-1} < \sqrt{n} = \lambda_n$.

Hereafter we may assume that $t \ge 1$. Thus $r(n) \le \sqrt{n-1} - \pi/8 + 1/\sqrt{n-1} \le \sqrt{s^2+t-1} - \pi/8 + 1/s \le s + t/(2s) - (1/(2s) + \pi/8 - 1/s) = s + t/(2s) + 1/(2s) - \pi/8$, and since $s \ge 4$ we see $1/(2s) - \pi/8 < -1/4$. Therefore $r(n) < \lambda_n$ follows if we show that $s + t/(2s) - 1/4 \le (s^2+t)s/(s^2+(t+1)/2)$, which simplifies to $2s^2 + t(t+1) \le s^3 + (t+1)s/2$. But using $t \le 2s$ and $t \ge 1$, respectively, we have $2s^2 + t(t+1) \le 6s^2 + 2s$ and $s^3 + s \le s^3 + (t+1)s/2$, so it is enough to show that $6s^2 + 2s \le s^3 + s$, which is true for $s \ge 7$.

Now (b) is clear: by Proposition 4.2(b) we have $R(m,n) \le mr(n) + 2n$, and we have just checked that $r(n) < \lambda_n$. ∎

5. NAGATA'S CONJECTURE

Nagata's conjecture, that $\delta(m,n) > m\sqrt{n}$, has been verified by [4] when n is a power of 4, and for various small m: by [1], [2] for $m < 13$ and $n > 9$, and for m up to about $\sqrt{n/2}$ by [3]. Moreover, examples suggest that $R(m,n) > m\sqrt{n}$ for m up to about $\sqrt{n}$. Our bound $\delta(m,n) \geq mnd/r$ from Proposition 3.1 also implies $\delta(m,n) > m\sqrt{n}$ in certain situations, one such we give here.

THEOREM 5.1. *Let $n = s^2+s$, where $s > 1$ is an integer. Then $\delta(m,n) > m\sqrt{n}$ holds for all $m \leq 2s$, and it holds for $m \leq 4s$ if m is even.*

Proof. By Proposition 3.1, $\delta(m,n) \geq \lceil mnd/r \rceil$ whenever $(r/d)^2 \geq n \geq r$, so for such an r and d it suffices to check that $\lceil mnd/r \rceil > m\sqrt{n}$ whenever $m \leq 2s$, or $m \leq 4s$ if m is even.

First assume that s is odd. By the remarks following Theorem 3.2, we may take $d = s-1$ and $r = s(s-1) + (s-1)/2$ (here is where we use $s > 1$). Then $mnd/r = mn/(s+1/2) = mn(s+1/2)/(s+1/2)^2 = mn(s+1/2)/(n+1/4)$ is less than $m(s+1/2)$ (in case m is even) and less than $m(s+1/2)+1/2$ (in case m is odd). It is not hard to check that mnd/r is greater than $m(s+1/2)-1$ (in case m is even and $m \leq 4s+1$) and greater than $m(s+1/2)+1/2-1$ (in case m is odd and $m \leq 2s$). It follows that $\lceil mnd/r \rceil$ is equal to $m(s+1/2)$ (in case m is even and $m \leq 4s+1$)) and to $m(s+1/2)+1/2$ (in case m is odd and $m \leq 2s$). But $m(s+1/2)$ (and obviously $m(s+1/2)+1/2$ too) is bigger than $m\sqrt{n}$, so, when s is odd, $\delta(m,n) > m\sqrt{n}$ holds for all $m \leq 2s$, and it holds for $m \leq 4s+1$ when m is even.

The case that s is even is the same, except this time we use the values of r and d which give λ_n; i.e., $r = s^2 + s/2$, $d = s$. Since $mnd/r = mn(s+1/2)/(n+1/4)$, as before, the rest of the proof proceeds unchanged. ∎

REMARK 5.2. It is interesting to recall an observation of Z. Ran, that a bound of the form $\delta(m,n) \geq cm$ for some constant $c > 0$ gives rise to a bound on regularity; i.e., on $\tau(m,n)$, the least degree t such that n general points of multiplicity m impose independent conditions on all forms of degree t or more. Ran has observed that $\tau(m,n) \leq (\max(\sqrt{n}, n/c))(m+1)-2$, and, if $c < \sqrt{n}$, $\tau(m,n) \leq \lceil n(m+1)/c \rceil - 3$. We thus obtain from Proposition 3.1 the bound $\tau(m,n) \leq \lceil (m+1)r/d \rceil - 3$, which for large m seems to be the best general bound now known. (If n is a square, one can do much better; see Lemma 5.3 of [9], which implies that $\tau(m,n) = m\sqrt{n} + \lceil (\sqrt{n}-3)/2 \rceil$ for m sufficiently large. On the other hand, for m small and n not a square, the bound $\tau(m,n) \leq d_1(m,n)$ given in [14] seems to be the best general bound currently available.)

Ran's observation works as follows. Let X be the surface obtained by blowing up the n general points, with $e_0, \ldots, e_n$ as defined in Section 2, and let $F = te_0 - m(e_1 + \cdots + e_n)$, where t satisfies both $t > \sqrt{n}(m+1) - 3$ and $t \geq (n/c)(m+1) - 3$. Now $(F - K_X)^2 > 0$ since $t > \sqrt{n}(m+1) - 3$, so $F - K_X$ is big. If $F - K_X$ were not nef, then there would be a reduced, irreducible curve C' with $(F - K_X) \cdot C' < 0$, and by averaging over the points, we would obtain from C' a curve C such that $0 > n(F - K_X) \cdot C' = (F - K_X) \cdot C$ and whose class is $be_0 - h(e_1 + \cdots + e_n)$ with $b \geq \delta(h, n) \geq ch$. But by hypothesis $t \geq (n/c)(m+1) - 3$, so $(F - K_X) \cdot C = (t+3)b - nh(m+1) \geq (n/c)(m+1)b - nh(m+1) \geq n(m+1)(b/c - h) \geq 0$. Thus $F - K_X$ is big and nef, so by Ramanujam or Kodaira vanishing (see Theorem 1.6 [15] for the characteristic p version, or see Theorem 2.8 of [7]) $0 = h^1(X, \mathcal{O}_X(K_X + F - K_X)) = h^1(X, \mathcal{O}_X(F))$; i.e., the points impose independent conditions. Finally, if $c < \sqrt{n}$, then $t \geq (n/c)(m+1) - 3$ implies $t > \sqrt{n}(m+1) - 3$.

Remark 5.3. In closing we mention that, given a reduced and irreducible plane curve of degree d through n general points of multiplicity m, Xu [16] shows (in characteristic 0) that $d > m\sqrt{n} - 1/(2\sqrt{n-1})$ and that $d \geq \sqrt{n-1}m$. For $n = s^2 + t$ with $0 \leq t \leq 2s$, we note in passing that $\lambda_n > \sqrt{n-1}$ if (but only if) t is even.

Note added in proof: see [17] for a generalization of Proposition 3.1.

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