On Nagata’s Conjecture*

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This paper gives an improved lower bound on the degrees \(d\) such that for general points \(p_1, \ldots, p_n \in \mathbb{P}^2\) and \(m > 0\) there is a plane curve of degree \(d\) vanishing at each \(p_i\) with multiplicity at least \(m\).

Key Words: multiple points, plane curves, fat points, Nagata’s conjecture, regularity.

1. INTRODUCTION

In this paper we work over an arbitrary algebraically closed field. For positive integers \(m\) and \(n\), define \(\delta(m, n)\) to be the least integer \(d\) such that for general points \(p_1, \ldots, p_n \in \mathbb{P}^2\) (i.e., for an open dense set, depending on \(m\) and \(n\), of \(n\) points of \(\mathbb{P}^2\)) there is a curve of degree \(d\) vanishing at each point \(p_i\) with multiplicity at least \(m\). For \(n \geq 10\), Nagata [11] conjectures that \(\delta(m, n) > m \sqrt{n}\), and proves this when \(n > 9\) is a square. (For \(n \leq 9\), applying methods of [12] it can be shown that \(\delta(m, n) = \lceil cnm \rceil\), where \(c_n = 1, 1, 1.5, 2, 2, 12/5, 21/8, 48/17\) and 3 for \(n = 1, \ldots, 9\), resp. Recall for any real number \(c\) that \(\lfloor c \rfloor\) is the greatest integer less than or equal to \(c\) and \(\lceil c \rceil\) is the least integer greater than or equal to \(c\); in particular, \(\lfloor c \rfloor \leq c \leq \lceil c \rceil\).

Clearly, if \(n' \leq n\), then \(\delta(m, n') \leq \delta(m, n)\), so we see from Nagata’s result above that \(\delta(m, n) \geq \delta(m, \lceil \sqrt{n} \rceil^2) > m \sqrt{n}\) for \(n \geq 16\). In fact, it is not hard to show directly for all \(n \geq 1\) the slightly weaker inequality \(\delta(m, n) \geq m \sqrt{n}\); similar reasoning shows \(\delta(m, n) \geq mn/\lceil \sqrt{n} \rceil\) as well (see

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Lemma 2.1). In certain ranges of \( n \), however, Roé [13] has recently given a better bound: for \( n \geq 3 \) he shows that \( \delta(m, n) \geq mr(n) \), where Roé’s constant \( r(n) \) is defined as \( r(n) = (n-1)\prod_{i=2}^{n-1}(1-i/(n-1+i^2)) \). Roé applies a procedure, called unloading, to an astute sequence of specializations, to derive an algorithm for computing a value \( R(m, n) \) depending on \( m \) and \( n \). It turns out on general principles that \( \delta(m, n) \geq R(m, n) \); the bound \( \delta(m, n) \geq mr(n) \) is obtained by showing that \( R(m, n) \geq mr(n) \).

Although it seems hard actually to prove that \( R(m, n) > m\sqrt{n} \) for \( m < \sqrt{n} \), examples suggest that this is at least typically true. Indeed, a direct check shows for \( 2 \leq m \leq 100 \) that \( R(m, m^2) \) is, plus or minus at most 1, equal to \( m^2 + m/10 \). (In a personal communication, Prof. Roé has told me that in fact \( R(m, m^2) \geq m^2 + |m/10| \) for \( m \) up to 200.) These and other data indicate that Roé’s result \( \delta(m, n) \geq R(m, n) \) is the best general bound currently known when \( m \) is not too large compared to \( n \). For comparison, [3] proves Nagata’s conjecture for values of \( m \) up to about \( \sqrt{n}/2 \), and, in characteristic 0, [4] determines \( \delta(m, n) \) for any \( m \) when \( n \) is a power of 4, while [1] and [2] do so for \( m \leq 12 \) and \( n \geq 10 \). These exact values agree with conjectures (see [5], [10], [6], [1], [2], [9]) which imply for \( n \geq 10 \) that \( \delta(m, n) \) should be the least positive integer \( d \) such that \( d^2 + 3d + 2 - nm^2 - mn > 0 \). When \( n \) is an even square and \( m \geq (\sqrt{n} - 2)/4 \), this \( d \) is precisely \( m\sqrt{n} + (\sqrt{n} - 2)/2 \) (see [9]), which unfortunately tends to be somewhat larger than \( R(m, n) \). (As an aside, we mention that the current paper resulted from this author’s wondering whether Roé’s algorithm might in some cases be used to justify \( \delta(m, n) = m\sqrt{n} + (\sqrt{n} - 2)/2 \) when \( n \) is an even square and \( m \geq (\sqrt{n} - 2)/4 \), in which case the results of [9] would determine a minimal free resolution of the \( m \)-th symbolic power \( I(m, n) \) of the ideal generated by all forms vanishing at \( n \) general points of \( \mathbb{P}^2 \).

As discussed in [9], to write down the resolution of \( I(m, n) \) it is enough to know two things: its Hilbert function \( h_{m,n}(t) = \dim I(m, n)_t \), which gives the dimension of the homogeneous component \( I(m, n)_t \) of \( I(m, n) \) in each degree \( t \), and the number \( \nu_t \) of generators of \( I(m, n) \) in each degree \( t \) in any minimal set of homogeneous generators for \( I(m, n) \). Bounds on \( \delta(m, n) \) give partial information on the resolution, since clearly \( h_{m,n}(t) = 0 = \nu_t \) for \( t < \delta(m, n) \), and \( \nu_{\delta(m,n)} = h_{m,n}(\delta(m, n)) \). The case that \( n \) is an even square bigger than 9 is especially interesting since then, for \( m \) sufficiently large, \( \delta(m, n) \) is expected to be \( m\sqrt{n} + (\sqrt{n} - 2)/2 \), and [9] determines \( h_{m,n}(t) \) for all \( t \geq m\sqrt{n} + (\sqrt{n} - 2)/2 \) and \( \nu_t \) for all \( t > m\sqrt{n} + (\sqrt{n} - 2)/2 \).

Unfortunately, except in special cases, when \( n > 9 \) is a square Nagata’s result showing \( \delta(m, n) \geq m\sqrt{n} + 1 \) remains the best bound known. Thus the results of this paper are of interest mainly when \( n \) is not a square.

In Section 2 we set up the context in which we will obtain our results, and we recall some previously known bounds on \( \delta(m, n) \). In Section 3, using a single specialization inspired by Roé’s, we obtain our main result,
Theorem 3.2, which shows that

$$\delta(m, n) \geq \lceil m\lambda_n \rceil,$$

where $\lambda_n$ denotes $n\lfloor \sqrt{n} \rfloor / \lceil \sqrt{n} \rceil$. In Section 4 we show that this is an improvement on the bounds previously known. In particular, we verify that:

- $\lambda_n \geq \lfloor \sqrt{n} \rfloor$ for all $n \geq 1$, with equality if and only if $n$ or $n - 1$ is a square;
- $\lambda_n \geq n / \lceil \sqrt{n} \rceil$ for all $n \geq 1$, with equality if and only if $n$, $n + 1$ or $n + 2$ is a square;
- $\lambda_n > r(n)$ for all $n \geq 3$; and
- for each $n \geq 3$, that $\lim_{m \to \infty} (m\lambda_n - R(m, n)) = \infty$.

In Section 5 we show for certain values of $n$ with $m$ not too large, that our bound implies Nagata’s conjecture.

2. BACKGROUND

We refer the reader to [8] for justification and amplification of the material in this section. Given essentially distinct points $p_1, \ldots, p_n \in \mathbb{P}^2$ (meaning for $i = 0, \ldots, n - 1$ that $p_{i+1} \in X_i$ where $X_0 = \mathbb{P}^2$ and $\pi_{i+1} : X_{i+1} \to X_i$ is the blow up of $p_{i+1}$), we will denote $X_n$ simply by $X$, with the morphism $\pi : X \to X_0$ being the composition $\pi_n \circ \cdots \circ \pi_1$ of the blow ups. The inverse image of $p_i$ with respect to $\pi_i$ is a divisor on $X_i$; the class of the total transform to $X$ of this divisor will be denoted $e_i$. The class of the total transform to $X$ of a line in $\mathbb{P}^2 = X_0$ will be denoted $e_0$. The divisor class group on $X$ is then freely generated by the classes $e_i, i = 0, \ldots, n$, with the intersection form being defined by $e_i \cdot e_j = 0$ for $i \neq j$, $e_0^2 = 1$ and $e_i^2 = -1$ for $i > 0$.

Define $\delta_X(m, n)$ to be the least $t$ such that $h^0(X, \mathcal{O}_X(te_0 - m(e_1 + \cdots + e_n))) > 0$. Then $\delta(m, n)$ is the maximum value of $\delta_X(m, n)$ over all essentially distinct sets of $n$ points of $\mathbb{P}^2$. (By semicontinuity, it follows that $\delta(m, n) = \delta_X(m, n)$ for a general set of distinct points $p_1, \ldots, p_n$.) To give a bound $\delta(m, n) \geq d$, it clearly suffices to find a $d$ and an $X$ for which we can check $\delta_X(m, n) \geq d$ (i.e., for which $h^0(X, \mathcal{O}_X((d-1)e_0 - m(e_1+\cdots+e_n))) = 0$). This follows, for example, if $X$ has a numerically effective (also called nef) divisor $C$ such that $C \cdot ((d-1)e_0 - m(e_1+\cdots+e_n)) < 0$. The following lemma, which is well-known, is, as we show, easy to prove this way. (The slightly stronger result $\delta(m, n) > m\lfloor \sqrt{n} \rfloor$ which follows from [11] as mentioned above, requires a related but somewhat more involved argument.)
Lemma 2.1. Let $m$ and $n$ be positive integers. Then we have:

(a) $\delta(m, n) \geq m[\sqrt{n}]$, and
(b) $\delta(m, n) \geq mn/[\sqrt{n}]$.

Proof. To prove (a), choose distinct points $p_1, \ldots, p_r$ of a smooth irreducible plane $r$-ic $C'$, with $r = [\sqrt{n}]$. Let $X$ be the surface obtained by blowing up $\mathbf{P}^2$ at $p_1, \ldots, p_r$ and let $C$ be the class of the proper transform to $X$ of $C'$. Then $C$ (being reduced and irreducible with $C^2 \geq 0$) is numerically effective; i.e., by definition $C \cdot F \geq 0$ for every class $F$ on $X$ with $h^0(X, \mathcal{O}_X(F)) > 0$ (we will refer to such a class $F$ as an effective class). In particular, $\delta(m, n)e_0 - m(e_1 + \cdots + e_r)$ is effective since $\delta(m, n) \geq \delta(m, r^2)$ and since $\delta(m, r^2)e_0 - m(e_1 + \cdots + e_r)$ is effective, so we have $\delta(m, n)r \geq mr^2$, and hence $\delta(m, n) \geq mr = m[\sqrt{n}]$.

To prove (b), choose distinct points $p_1, \ldots, p_n$ of a smooth irreducible plane $r$-ic $C''$, where this time $r = \lceil \sqrt{n} \rceil$ and $X$ is the surface obtained by blowing up $\mathbf{P}^2$ at $p_1, \ldots, p_n$ and $C$ is the class of the proper transform to $X$ of $C''$. Then reasoning as above gives $\delta(m, n)r \geq mn$, and hence the result.

3. THE MAIN RESULT

In this section, we use a special arrangement of essentially distinct points, similar to what is used in [13], to which we will apply an argument analogous to that used in the proof of Lemma 2.1.

Proposition 3.1. Let $d, n$ and $r$ be positive integers such that $(r/d)^2 \geq n$ and $r \leq n$. Then $\delta(m, n) \geq mn/r$.

Proof. Let $C_1$ be a smooth plane curve of degree $d$. Choose any point $p$ such that $p \in C_1 \subset X_0 = \mathbf{P}^2$. Let $X_1$ be the blow up of $X_0$ at $p_1$, and let $C_2$ be the proper transform of $C_1$. Then choose $p_2$ to be the point of $C_2$ infinitely near to $p_1$. Continue in this way, iteratively obtaining essentially distinct points $p_i$, $i = 1, \ldots, r$, where, for $1 < i \leq r$, $p_i$ is the point of $C_i$ infinitely near to $p_{i-1}$ with respect to the blowing up $\pi_i : X_{i-1} \to X_{i-2}$ of $p_{i-1}$, with $C_i$ being the proper transform of $C_{i-1}$ with respect to $\pi_{i-1}$. (Thus $C_1$ and $p_1$ determine $p_i$ for $1 < i \leq r$.)

If $n > r$, choose additional points $p_{r+1}, \ldots, p_n$ so that again each point $p_i$ is infinitely near to $p_{i-1}$ for $i \geq r + 1$ but so that $p_{r+1}$ is not on the proper transform of $C$, and none of $p_i$, $i \geq r + 1$ is on the proper transform to $X_{i-1}$ of the exceptional locus of the blow up morphism $X_{i-2} \to X_{i-3}$ (i.e., $p_i$ is chosen so that $e_{i-1} - e_i$ but not $e_{i-2} - e_{i-1} - e_i$ is effective). As
usual, we denote $X_n$ by $X$; $C$ will denote the class of the proper transform of $C_1$ to $X$.

Then $C$ is the class of a smooth, irreducible curve, as is each of $e_1 - e_2, \ldots, e_{n-1} - e_n$ and $e_n$. By hypothesis, $d^2n \leq r^2$ and $r \leq n$ and hence $d^2 \leq r$; using $d^2 \leq r$, it is not hard to verify that $rde_0 - d^2(e_1 + \cdots + e_r) - (r^2 - rd^2)e_{r+1}$ is the sum of $rC$ and various nonnegative multiples of $e_i - e_{i+1}$ for $1 \leq i \leq r$ (here we have assumed that $r < n$; we leave it to the reader to consider the case that $r = n$). But $d^2n \leq r^2$ implies $r^2 - rd^2 \geq (n-r)d^2$, hence the class $D = rde_0 - d^2(e_1 + \cdots + e_n)$ is the sum of $rde_0 - d^2(e_1 + \cdots + e_r) - (r^2 - rd^2)e_{r+1}$ and various nonnegative multiples of $e_n$ and of $e_i - e_{i+1}$ for $r+1 \leq i \leq n - 1$. Thus $D$ is effective. But $D \cdot C = 0$, $D \cdot (e_i - e_{i+1}) = 0$ for $i > 0$ and $D \cdot e_n \geq 0$, so $D$, being a sum of effective classes which it meets nonnegatively, is nef. Therefore, $\delta(m, n)rd - d^2mn = (\delta(m, n)e_0 - m(e_1 + \cdots + e_n)) \cdot D \geq 0$; i.e., $\delta(m, n) \geq mn/dr$, as claimed. [1]

As a corollary we derive:

**Theorem 3.2.** Let $n$ and $m$ be positive integers; then $\delta(m, n) \geq [m\lambda_n]$.

**Proof.** Apply Proposition 3.1 with $d = [\sqrt{n}]$ and $r = [\sqrt{md}]$. We merely need to check that $(r/d)^2 \geq n$ and $r \leq n$. Clearly, $(r/d)^2 \geq (\sqrt{md}/d)^2 = n$. For the other inequality, we have $\sqrt{n} = \sqrt{m\sqrt{n}} \leq \sqrt{n}\sqrt{\sqrt{n}} = n$, hence, since $n$ is an integer, $r = [\sqrt{n}\sqrt{\sqrt{n}}] \leq n$, as required. [1]

It does not seem easy to know a priori what choice of $r$ and $d$ is best. We can always write any given positive integer $n$ as $n = s^2 + t$, where $s = [\sqrt{n}]$ and $t = n - [\sqrt{n}]^2$, hence $t$ is between $0$ and $2s$. The choice $d = [\sqrt{n}]$ and $r = [\sqrt{n}\sqrt{\sqrt{n}}]$, which gives $\lambda_n$, amounts to taking $d = s$ and $r = s^2 + [t/2]$ (see the proof of Proposition 4.1), but this is not always the best choice. For example, if $1 < s \leq t < 2s - 1$ with $t$ odd, it is easy to check that $d = s - 1$ and $r = s(s - 1) + (t-1)/2$ satisfy the hypotheses of Proposition 3.1 but give a slightly better bound on $\delta(m, n)$ than does $\lambda_n$. However, in terms of both simplicity and general applicability, the author knows of no better choice than $d = [\sqrt{n}]$ and $r = [\sqrt{n}\sqrt{\sqrt{n}}]$.

4. COMPARISONS

We begin by comparing $\lambda_n$ with $[\sqrt{n}]$ and $n/[\sqrt{n}]$. We will use repeatedly the easy fact that any integer $n \geq 0$ can be (uniquely) written in the form $n = s^2 + t$, where $s$ is a nonnegative integer and $0 \leq t \leq 2s$ (indeed, $s$ is just $[\sqrt{n}]$).

**Proposition 4.1.** Let $n = s^2 + t$ where $s > 0$ and $0 \leq t \leq 2s$ are integers; then:
(a) \( \lambda_n \geq \lfloor \sqrt{n} \rfloor \), with equality if and only if \( t = 0 \) or \( t = 1 \), and
(b) \( \lambda_n \geq n/\lfloor \sqrt{n} \rfloor \), with equality if and only if \( t = 0 \), \( t = 2s - 1 \) or \( t = 2s \).

Proof. First, one sees that \( \lfloor \sqrt{n} \rfloor = s \). With slightly more trouble, one also checks that \( \lfloor \sqrt{n} \rfloor = s \). It is now easy to see that \( (s^2 + t)s/(s^2 + [t/2]) \geq s \), with equality if and only if \( t = 0 \) or \( 1 \); this is (a). Next, if \( t = 0 \), (b) is true, while for \( t > 0 \), we have \( (s^2 + t)s/(s^2 + [t/2]) \geq (s^2 + t)/(s + 1) \), with equality if and only if \( t \) is \( 2s \) or \( 2s - 1 \), which is the rest of (b).

We next want to compare \( m\lambda_n \) with Rœ’s bounds \( mr(n) \) and \( R(m, n) \). In order to deal with \( R(m, n) \) it will be helpful to describe Rœ’s algorithm for computing it.

We first develop some notation and terminology. Let \( w = (m_1, \ldots, m_n) \) be a vector; then \( p(w) \) will denote the vector obtained from \( w \) by putting the entries \( m_j \) with \( j > 1 \) into nonincreasing order. We will use \( v_i \) to denote the vector \((1, -1, \ldots, -1, 0, \ldots, 0)\), where there are \( i \) entries of \(-1\). Replacing every negative entry of a vector by \( 0 \) we will call rectification. We will define \( q_i(w) \) to be \( w \), if, with respect to the usual dot product, \( w \cdot v_i \geq 0 \); otherwise \( q_i(w) \) will be the rectification of \( w + v_i \).

Now let \( n \geq 3 \) be an integer; for each integer \( i \) with \( 2 \leq i \leq n - 1 \) we describe a routine \( O_i \). Given a vector \( w = (m_1, \ldots, m_n) \) of nonnegative integers, let \( g_i \) denote the composition \( pq_i \), so \( g_i(w) = pq_i(p(w)) \), and consider the sequence \( g_1(w), g_2(w), \ldots \). It is easy to see that eventually the sequence stabilizes at some vector which we will denote by \( O_i(w) \).

Rœ’s algorithm, then, is to apply the routines \( O_2, \ldots, O_{n-1} \) consecutively to an initial input vector \( w = (m, \ldots, m) \); the value \( R(m, n) \) is the first entry of the vector \( O_{n-1} \cdots O_2(w) \).

Since \( w = (m, \ldots, m) \) is of particular interest, in this case we will denote the first component of \( O_i \cdots O_2(w) \) by \( R_i(m, n) \) and set \( R_i(m, n) = m \); thus \( R_{n-1}(m, n) = R(m, n) \). The sum of the second through \( n \)th components of \( O_i \cdots O_2(w) \) will be denoted by \( S_i(m, n) \) and we set \( S_1(m, n) = (n - 1)m \). Suppressing \( m \) and \( n \) when no confusion will result, we may write \( R_i \) or \( S_i \) instead of \( R_i(m, n) \) or \( S_i(m, n) \).

Another description of the algorithm will be helpful. Given integers \( n > 1 \) and \( 1 \leq c \leq n - 1 \), we will use \( v(a, b, c, n) \) to denote the vector \((a, b, \ldots, b, b - 1, \ldots, b - 1)\), where there are \( c \) of the \( b \) entries and \( n \) entries altogether. For example, if \( w = (m, \ldots, m) \), then \( w = v(m, m, n - 1, n) \).

Now let \( v = v(m_1, b, c, n) \) with \( b \geq 1 \) and assume that \( v \cdot v_{i-1} \geq 0 \). Then \( g_i(v) = v \) if \( v \cdot v_i \geq 0 \) but if \( v \cdot v_i < 0 \) then \( g_i(v) \) is \( v(m_1 + 1, b', c', n) \), where \( b' = b \) if \( i < c \), in which case \( c' = c - i \), and \( b' = b - 1 \) if \( i \geq c \), in which case \( c' = n - i + c \). (Because of possible rectification, the case that
\[ b = 1 \text{ is a bit tricky, but since } \mathbf{v} \cdot \mathbf{v}_{i-1} \geq 0, \text{ if } b = 1 \text{ and } \mathbf{v} \cdot \mathbf{v}_i < 0, \text{ then } i \leq c, \text{ so no negative entries are ever involved.} \]

In Roe’s algorithm, we continue to apply \( g_i \) until the dot product with \( \mathbf{v}_i \) becomes nonnegative. If starting with \( \mathbf{v} \), \( t \) is the least number of such applications required for the dot product with \( \mathbf{v}_i \) to become nonnegative, then (denoting by \( S \) the sum \((n - 1)(b - 1) + c\) of all components of \( \mathbf{v} \) but the first) the result of applying \( g_i \) for \( j \leq t \) times is \( v(m_1 + j, [(S - ji)/(n - 1)], \rho_j, n) \), where \( \rho_j = (S - ji) - (n - 1)(S - ji)/(n - 1) \) is the remainder when \( S - ji \) is divided by \( n - 1 \). Looking at \( \mathbf{v} \cdot v(m_1 + j, [(S - ji)/(n - 1)], \rho_j, n) \) we see that \( t \) is the least integer \( j \) such that \( i[(S - ji)/(n - 1)] + \min(\rho_j, i) \leq m_1 + j \leq i[(S - (j - 1)i)/(n - 1)] + \min(\rho_{j-1}, i) \).

In particular, applying the above remarks to

\[
v(R_{i-1}, b, c, n) = \mathcal{O}_{i-1} \cdots \mathcal{O}_2(\mathbf{w})
\]

and \( v(R_i, b', c', n) = \mathcal{O}_i \cdots \mathcal{O}_2(\mathbf{w}) \), where \( \mathbf{w} = (m, \ldots, m) \), we have the following formulas:

(F1) \( R_i = R_{i-1} + t \) where \( t \) is the least \( j \) such that \( i[(S_{i-1} - ji)/(n - 1)] + \min(\rho_j, i) \leq R_{i-1} + j \leq i[(S_{i-1} - (j - 1)i)/(n - 1)] + \min(\rho_{j-1}, i) \);

(F2) \( S_{i-1} = (b - 1)(n - 1) + c \); and

(F3) \( S_i = S_{i-1} - i(R_i - R_{i-1}) \) or equivalently \( R_i + S_i/i = R_{i-1} + S_{i-1}/i \) (which are the same as \( i(R_i - R_{i-1}) = S_{i-1} - S_i \), which holds since \( R \) increases by \( 1 \) for each decrease in \( S \) by \( i \)).

We note that the value \( mr(n) \) can be obtained by a similar but “averaged” procedure, which requires working over the rationals. In place of \( \mathbf{v}_i \) we have \( \mathbf{v}_i = (1, -i/(n - 1), \ldots, -i/(n - 1)) \), and in place of \( q_i \) we have \( \mathbf{q}_i \), where \( \mathbf{q}_i(\mathbf{w}) \) is \( \mathbf{w} \) if \( \mathbf{w} \cdot \mathbf{v}_i \geq 0 \); otherwise \( \mathbf{q}_i(\mathbf{w}) \) is the rectification of \( \mathbf{w} + i\mathbf{v}_i \), where \( t \) is chosen so that \( (\mathbf{w} + i\mathbf{v}_i) \cdot \mathbf{v}_i = 0 \). We define \( \mathbf{q}_i \) to be \( p\mathbf{q}_i \) (we use \( p \) simply for analogy; because of the averaging, nothing important would be affected if we did not use it), and we take \( \overline{\mathbf{v}}(\mathbf{w}) \) to be the vector at which the sequence \( \mathbf{q}_i(\mathbf{w}), \ldots, \mathbf{q}_i(\mathbf{w}) \) stabilizes. (Note that \( \overline{\mathcal{O}}_i(\mathbf{w}) = \mathbf{q}_i(\mathbf{w}) \) if neither \( \mathbf{w} \) nor \( \mathbf{q}_i(\mathbf{w}) \) has a negative entry.)

Now let \( m \) be a positive integer, let \( r_1(n) = 1 \), let \( mr_i(n) \) be the first entry of \( \mathbf{w}_i = \overline{\mathcal{O}}_1 \cdots \overline{\mathcal{O}}_m(\mathbf{w}_1) \), where \( \mathbf{w}_1 \) is the \( n \)-vector \((m, \ldots, m) \), let \( s_1(n) = n - 1 \) and let \( ms_i(n) \) be the sum of all of the entries but the first of \( \mathbf{w}_i \). It is not hard by induction to check that

(f0) \( \mathbf{w}_i \cdot \mathbf{v}_i = 0 \) for \( 2 \leq i \leq n - 1 \) and hence \( \mathbf{w}_i \cdot \mathbf{v}_{i+1} < 0 \) for \( 1 \leq i < n - 1 \), and that

(f1) \( r_1(n) = r_{i-1}(n)(i/(i - 1))(1 - i/(n - 1 + i^2)), \) and

(f2) \( s_i(n) = ((n - 1)/i)r_i(n) \), from which it follows that

(f3) \( r_i(n) + s_i(n)/i = r_{i-1}(n) + s_{i-1}(n)/i \) and that
(f4) \( r_i(n) = (i^2/(n-1+i^2))(r_{i-1} + s_{i-1}/i) \).

By (f1), of course, we have \( r_i(n) = \Pi_{j=2}^{i}(j(1-j/(n-1+j^2))/(j-1)) \) and hence \( r(n) = r_{n-1}(n) \).

**Proposition 4.2.** Let \( n \geq 3 \) be an integer; then:

(a) \( r(n) \leq \sqrt{n-1} - \pi/8 + 1/\sqrt{n-1} \), and
(b) \( R(m,n) \leq mr(n) + 2(n-1) \).

**Proof.** On behalf of easier reading, we will in this proof use \( k \) to denote \( n-1 \). By direct check, (a) holds for \( 2 \leq k \leq 3 \). So assume \( k \geq 4 \). In any case, \( r(n) = k\Pi_{i=2}^{k}(1 - i/(k+i^2)) \), and [13] shows that \( (r(n))^2 = k\Pi_{i=1}^{k-1}(1 - (i/(k+i^2))^2) \). But \( \log(1-x) < -x \) holds for \( 0 < x < 1 \), so we have \( \sum_{i=1}^{k-1} \log(1 - (i/(k+i^2))) \leq -\sum_{i=1}^{k-1}(i/(k+i^2))^2 \). However, \( \sum_{i=k}^{\infty}(i/(k+i^2))^2 \leq \int_{i=k}^{\infty} x/(x^2 + k^2) \leq 1/(k-1) \), and from [13] we see that \( \sum_{i=1}^{k-1}(i/(k+i^2))^2 = \pi(1 - \pi^2 + 4\sinh^2(\sqrt{k}\pi))/4\sinh^2(\sqrt{k}\pi) - (\pi/(4\sqrt{k})) \) equals \( \pi/(4\sqrt{k}) \) (which is at least as big as \( \pi/(4\sqrt{k}) \)). But \( \sum_{i=1}^{k-1}(i/(k+i^2))^2 \leq 1/(k-1) + \pi^2/(4\sinh^2(\sqrt{k}\pi)) - (\pi/(4\sqrt{k})) \) (which is at least as big as \( \pi/(4\sqrt{k}) \)). Hence \( r(n) \leq \sqrt{k}(1 + 2/k - \pi/(4\sqrt{k}) \) where

\[
\frac{\pi^2}{4\sinh^2(\sqrt{k}\pi)} < 0.1/\sqrt{k-1}
\]

follows from \( (10n^2/4)(k-1) < \sinh^2(\sqrt{k}\pi) \), which itself is easy to check (look at a graph first). Since \( 1.1/(k-1) - \pi/(4\sqrt{k}) \) is negative for \( k \geq 4 \), the Taylor series for \( \exp(1.1/(k-1) - \pi/(4\sqrt{k})) \) is alternating so \( (r(n))^2 \leq k\exp(1.1/(k-1) - \pi/(4\sqrt{k})) \leq k(1 + 1.1/(k-1) - \pi/(4\sqrt{k})) + 1/2(1.1/(k-1) - \pi/(4\sqrt{k}))^2) \), but \( 1.1/(k-1) + 1/2(1.1/(k-1) - \pi/(4\sqrt{k}))^2 < 1.1/(k-1) + 1/2(1/2) \). Hence \( r(n) \leq \sqrt{k}(1 + 2/k - \pi/(4\sqrt{k}) \) where

Now consider (b). We begin by showing \( R_i \leq (i^2R_{i-1} + iS_{i-1} + i^2 + \sqrt{k})/(k+i^2 + k) \). Let \( t = R_{i-1} - R_{i-1} \). By (F1), we have \( R_{i-1} + t \leq i(\min(p_{n-1}, k) + (S_{i-1} - (t-1)i))/k + i \) and hence \( R_{i-1} + t \leq i(\min(p_{n-1}, k) + (S_{i-1} - (t-1)i))/k + i \). But \( i(\min(p_{n-1}, k) + (S_{i-1} - (t-1)i))/k + i \leq i(S_{i-1} - ti)/k + (i^2 + ki)/k \), so solving for \( t \)
gives \( t \leq (iS_{i-1} - kR_{i-1} + i^2 + ki)/(k + i^2) \) and therefore \( R_t = R_{t-1} + t \leq (iS_{i-1} + i^2 R_{i-1} + i^2 + ki)/(k + i^2) \), as claimed.

Now (given \( m \) and \( n \), and suppressing the \( n \) notationally) it will be sufficient to prove by induction for each \( i \) that \( R_i \leq mr_i + 2k \) and \( R_i + S_i/i \leq mr_i + s_i/i + 2k \). Note that \( R_1 = m \leq m + 2k = mr_1 + 2k \), and \( R_1 + S_1/i = nm \leq nm + 2k = mr_1 + ms_1/i + 2k \). So assume that \( R_{i-1} \leq mr_{i-1} + 2k \) and \( R_{i-1} + S_{i-1}/(i-1) \leq mr_{i-1} + ms_{i-1}/(i-1) + 2k \) hold for some \( i \geq 2 \).

Since \( R_{i-1} \leq mr_{i-1} + 2k \) and \( R_{i-1} + S_{i-1}/(i-1) \leq mr_{i-1} + ms_{i-1}/(i-1) + 2k \), then \( R_i + S_i/i \leq mr_{i-1} + ms_{i-1}/i + 2k \) must also hold, and using (F3) and (3) we therefore have \( R_i + S_i/i = R_{i-1} + S_{i-1}/i \leq mr_{i-1} + ms_{i-1}/i + 2k \), as required.

Since \( R_{i-1} + S_{i-1}/i \leq mr_{i-1} + ms_{i-1}/i + 2k \) and \( R_i \leq (iS_{i-1} + i^2 R_{i-1} + i^2 + ki)/(k + i^2) = (i^2/(k+i^2))(R_{i-1} + S_{i-1}/i)/(i^2 + ki) \), the latter is at most \((i^2/(k+i^2))(mr_{i-1} + ms_{i-1}/i + (2ki^2 + i^2 + ki)/(i^2 + k)) \), so by (F4), this latter simplifies to \( mr_i + (2ki^2 + i^2 + ki)/(i^2 + k) \) which (taking \( k \) to be \( i \)) is at most \( mr_i + (2k^3 + 2k^2)/(k^2 + k) = mr_i + 2k \), as we needed to show.

We now compare our bound with those of Roé.

**Proposition 4.3.** Let \( n \geq 3 \) be an integer; then:

(a) \( \lambda_n > r(n) \), and

(b) \( \lim_{m \to \infty} m\lambda_n - R(m, n) = \infty \). In particular, \( [m\lambda_n] > R(m, n) \) for all sufficiently large \( m \).

**Proof.** Let \( s = \lfloor \sqrt{n} \rfloor \) and write \( n = s^2 + t \); thus \( 0 \leq t \leq 2s \) and \( \lambda_n = (s^2 + t)s/[s\sqrt{n}] \geq (s^2 + t)s/[s(s + t/2s)] = (s^2 + t)s/[s^2 + t/2] \geq (s^2 + t)s/(s^2 + t + 1)/2 \).

For part (a), one first checks case by case that \( r(n) < \lambda_n \) for \( 3 \leq n \leq 48 \), so we are reduced to the case that \( n \geq 49 \); i.e., \( s \geq 7 \). First assume \( t = 0 \); then in fact \( \lambda_n = \sqrt{n} \). But since \( n \geq 8 \) we see \(-\pi/8 + 1/\sqrt{n-1} < 0 \) so, by Proposition 4.2(a), \( r(n) \leq \sqrt{n-1} - \pi/8 + 1/\sqrt{n-1} < \sqrt{n-1} = \lambda_n \).

Hereafter we may assume that \( t \geq 1 \). Thus \( r(n) \leq \sqrt{n-1} - \pi/8 + 1/\sqrt{n-1} \leq \sqrt{n-1} - \pi/8 + 1/\sqrt{n-1} \leq \sqrt{s^2 + t + 1} - \pi/8 + 1/\sqrt{s^2 + t + 1}/s \leq s + t/(2s) - (1/(2s) + \pi/8 - 1/s) = s + t/(2s) + 1/(2s) - \pi/8 \), and since \( s \geq 4 \) we see \( 1/(2s) - \pi/8 < -1/4 \). Therefore \( r(n) < \lambda_n \) follows if we show that \( s + t/(2s) - 1/4 \leq (s^2 + t)s/[s^2 + (t+1)/2] \), which simplifies to \( 2s^2 + t(t+1) \leq s^3 + (t+1)s/2 \). But using \( t \leq 2s \) and \( t = 1 \), respectively, we have \( 2s^2 + t(t+1) \leq 6s^2 + 2s \) and \( s^3 + s \leq s^3 + (t+1)s/2 \), so it is enough to show that \( 6s^2 + 2s \leq s^3 + s \), which is true for \( s \geq 7 \).

Now (b) is clear: by Proposition 4.2(b) we have \( R(m, n) \leq mr(n) + 2n \), and we have just checked that \( r(n) < \lambda_n \).
5. NAGATA’S CONJECTURE

Nagata’s conjecture, that $\delta(m, n) > m\sqrt{n}$, has been verified by [4] when $n$ is a power of 4, and for various small $m$: by [1], [2] for $m < 13$ and $n > 9$, and for $m$ up to about $\sqrt{n}/2$ by [3]. Moreover, examples suggest that $R(m, n) > m\sqrt{n}$ for $m$ up to about $\sqrt{n}$. Our bound $\delta(m, n) \geq m\sqrt{n}$ from Proposition 3.1 also implies $\delta(m, n) > m\sqrt{n}$ in certain situations, one such we give here.

**Theorem 5.1.** Let $n = s^2 + s$, where $s > 1$ is an integer. Then $\delta(m, n) > m\sqrt{n}$ holds for all $m \leq 2s$, and it holds for $m \leq 4s$ if $m$ is even.

**Proof.** By Proposition 3.1, $\delta(m, n) \geq \lceil mnd/r \rceil$ whenever $(r/d)^2 \geq n \geq r$, so for such an $r$ and $d$ it suffices to check that $\lceil mnd/r \rceil > m\sqrt{n}$ whenever $m \leq 2s$, or $m \leq 4s$ if $m$ is even.

First assume that $s$ is odd. By the remarks following Theorem 3.2, we may take $d = s - 1$ and $r = s(s - 1) + (s - 1)/2$ (here is where we use $s > 1$). Then $mnd/r = mn/(s + 1/2) = mn(s + 1/2)/(s + 1/2) = mn(s + 1/2)/(n + 1/4)$ is less than $m(s + 1/2)$ (in case $m$ is even) and less than $m(s + 1/2) + 1/2$ (in case $m$ is odd). It is not hard to check that $mnd/r$ is greater than $m(s + 1/2) - 1$ (in case $m$ is even and $m \leq 4s + 1$) and greater than $m(s + 1/2) + 1/2 - 1$ (in case $m$ is odd and $m \leq 2s$). It follows that $\lceil mnd/r \rceil$ is equal to $m(s + 1/2)$ (in case $m$ is even and $m \leq 4s + 1$) and to $m(s + 1/2) + 1/2$ (in case $m$ is odd and $m \leq 2s$). But $m(s + 1/2)$ (and obviously $m(s + 1/2) + 1/2$ too) is bigger than $m\sqrt{n}$, so, when $s$ is odd, $\delta(m, n) > m\sqrt{n}$ holds for all $m \leq 2s$, and it holds for $m \leq 4s + 1$ when $m$ is even.

The case that $s$ is even is the same, except this time we use the values of $r$ and $d$ which give $\lambda_n$: i.e., $r = s^2 + s/2, d = s$. Since $mnd/r = mn(s + 1/2)/(n + 1/4)$, as before, the rest of the proof proceeds unchanged. 

**Remark 5.2.** It is interesting to recall an observation of Z. Ran, that a bound of the form $\delta(m, n) \geq cm$ for some constant $c > 0$ gives rise to a bound on regularity; i.e., on $\tau(m, n)$, the least degree $t$ such that $n$ general points of multiplicity $m$ impose independent conditions on all forms of degree $t$ or more. Ran has observed that $\tau(m, n) \leq \lceil (\sqrt{n}, n/c) \rceil (m + 1) - 2$, and, if $c < \sqrt{n}, \tau(m, n) \leq \lceil n(m + 1)/c \rceil - 3$. We thus obtain from Proposition 3.1 the bound $\tau(m, n) \leq \lceil (m + 1)r/d \rceil - 3$, which for large $m$ seems to be the best general bound now known. (If $n$ is a square, one can do much better; see Lemma 5.3 of [9], which implies that $\tau(m, n) = m\sqrt{n} + \lceil (\sqrt{n} - 3)/2 \rceil$ for $m$ sufficiently large. On the other hand, for $m$ small and $n$ not a square, the bound $\tau(m, n) \leq d_1(m, n)$ given in [14] seems to be the best general bound currently available.)
Ran’s observation works as follows. Let \( X \) be the surface obtained by blowing up the \( n \) general points, with \( e_0, \ldots, e_n \) as defined in Section 2, and let \( F = te_0 - m(e_1 + \cdots + e_n) \), where \( t \) satisfies both \( t > \sqrt{n(m+1) - 3} \) and \( t \geq (n/c)(m+1) - 3 \). Now \((F - K_X)^2 > 0\) since \( t > \sqrt{n(m+1) - 3} \), so \( F - K_X \) is big. If \( F - K_X \) were not nef, then there would be a reduced, irreducible curve \( C' \) with \((F - K_X) \cdot C' < 0\), and by averaging over the points, we would obtain from \( C' \) a curve \( C \) such that \( 0 > n(F - K_X) \cdot C = (F - K_X) \cdot C \) and whose class is \( b - h(e_1 + \cdots + e_n) \) with \( b \geq \delta(h, n) \geq ch \). But by hypothesis \( t \geq (n/c)(m+1) - 3 \), so \((F - K_X) \cdot C = (t + 3)b - nh(m + 1) \geq (n/c)(m+1)b - nh(m+1) \geq n(m+1)(b/c - h) \geq 0 \). Thus \( F - K_X \) is big and nef, so by Ramanujam or Kodaira vanishing (see Theorem 1.6 [15] for the characteristic \( p \) version, or see Theorem 2.8 of [7]) \( 0 = h^1(X, \mathcal{O}_X(F + F - K_X)) = h^1(X, \mathcal{O}_X(F)) \); i.e., the points impose independent conditions. Finally, if \( c < \sqrt{n} \), then \( t \geq (n/c)(m+1) - 3 \) implies \( t > \sqrt{n(m+1) - 3} \).

**Remark 5.3.** In closing we mention that, given a reduced and irreducible plane curve of degree \( d \) through \( n \) general points of multiplicity \( m \), Xu [16] shows (in characteristic 0) that \( d > m \sqrt{n - 1}/(2 \sqrt{n - 1}) \) and that \( d \geq \sqrt{n - 1}m \). For \( n = s^2 + t \) with \( 0 \leq t \leq 2s \), we note in passing that \( \lambda_n > \sqrt{n - 1}t \) if (but only if) \( t \) is even.

Note added in proof: see [17] for a generalization of Proposition 3.1.

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