Homework 1: Math 953 Spring 2005

Due January 21, 2005

Let $\mathcal{C}$ and $\mathcal{D}$ be categories.

Definition: A (covariant) *functor* $F : \mathcal{C} \to \mathcal{D}$ is an assignment of an object $F(A)$ of $\mathcal{D}$ for each object A of $\mathcal{C}$ and a collection of maps $F : \mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$ for each pair of objects A and B of $\mathcal{C}$, such that $F(1_A) = 1_{F(A)}$ and whenever $f \circ g$ is defined, $F(f \circ g) = F(f) \circ F(g)$.

Definition: A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ is an assignment of an object $F(A)$ of $\mathcal{D}$ for each object A of $\mathcal{C}$ and a collection of maps $F : \mathrm{Hom}(A, B) \to \mathrm{Hom}(F(B), F(A))$ for each pair of objects A and B of $\mathcal{C}$, such that $F(1_A) = 1_{F(A)}$ and whenever $f \circ g$ is defined, $F(f \circ g) = F(g) \circ F(f)$. (Alternatively, a contravariant functor $F : \mathcal{C} \to \mathcal{D}$ is just a functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.)

Definition: A functor $F : \mathcal{C} \to \mathcal{D}$ is an *isomorphism* of categories if there is a functor $G : \mathcal{D} \to \mathcal{C}$ such that FG is the identity functor on $\mathcal{D}$ and GF is the identity functor on $\mathcal{C}$.

Definition: Let $f \in \mathrm{Hom}(B, A)$ be an arrow in a category $\mathcal{C}$. We say that f is a *monomorphism* if for every object C, there never are arrows $g, h \in \mathrm{Hom}(C, B)$ such that $fg = fh$ but $g \neq h$.

(1) Let G and H be groups and let $\mathcal{G}$ ($\mathcal{H}$, resp.) be the category with a single object whose arrows are the elements of G (H, resp.), with the group law giving composition of arrows. Show that there is a bijection between functors $F : \mathcal{G} \to \mathcal{H}$ and homomorphisms $f : G \to H$, and that F is an isomorphism if and only if its corresponding homomorphism f is.

(2) Let X and Y be topological spaces and let $\mathcal{TOP}(X)$ and $\mathcal{TOP}(Y)$ be the corresponding categories. For each continuous map $f : X \to Y$ define a functor $F_f : \mathcal{TOP}(Y) \to \mathcal{TOP}(X)$ such that F_f is an isomorphism if f is a homeomorphism. If F_f is an isomorphism, must f be a homeomorphism? Is every functor $F : \mathcal{TOP}(Y) \to \mathcal{TOP}(X)$ of the form F_f for some f?

(3) Show that a monomorphism in the category $\mathcal{SET}$ of sets is the same thing as an injective map.

(4) Define *epimorphism* in an arbitrary category such that an epimorphism in the category $\mathcal{SET}$ of sets is the same thing as a surjective map. Compare epimorphisms in a category $\mathcal{C}$ with monomorphisms in $\mathcal{C}^{op}$.