M918: Algebraic Curves Problem Assignment 3, due Friday, October 2, 2009.

(Do the problems appropriate for your background or those for which you'd like feedback on.)

One version of Hilbert's Nullstellensatz is: If $I \subseteq \mathbf{C}[x_0, \ldots, x_n]$ is an ideal generated by homogeneous elements, and if $G \in \mathbf{C}[x_0, \ldots, x_n]$ is homogeneous and nonconstant such that $V(I) \subseteq V(G) \subseteq \mathbf{P}^n$, then $G \in \sqrt{I}$ (i.e., a power of $G$ is in $I$).

[1] If $F$ and $G$ are nonconstant homogeneous polynomials in $\mathbf{C}[x_0, \ldots, x_2]$ such that $F$ is irreducible and $V(F) \subseteq V(G) \subseteq \mathbf{P}^2$, give two proofs that $F$ divides $G$ (one using Bézout, and the other using the Nullstellensatz).

[2] Assume $F$ and $G$ are nonconstant polynomials in $\mathbf{C}[x_0, \ldots, x_n]$ such that $F$ divides $G$. If $G$ is homogeneous, prove that $F$ is too.

[3] Let $C = V(F)$ be a curve defined by some nonconstant square-free homogeneous $F \in \mathbf{C}[x, y, z]$ (i.e., $F$ is not divisible by the square of any irreducible polynomial). If $C \subset \mathbf{P}^2$ is a smooth curve, prove that $F$ is irreducible. (Hint: If $F$ factors, show that any point $p$ where both factors vanish is a singular point.)

[4] Show that there is no smooth plane curve $C$ in $\mathbf{P}^2$ of genus 2.

[5] Show that a smooth cubic curve $C$ in $\mathbf{P}^2$ has at most 9 flex points (but do not use the group law on $C$).

[6] Show that there are at most 12 lines through pairs of flex points of a smooth cubic curve $C$ in $\mathbf{P}^2$. (Hint: use the fact that a line through any two flex points goes through a third flex point.) Look up and use the Sylvester-Gallai theorem to conclude that over the complex numbers, if $C$ is defined by a form with real coefficients, then at most 3 of the flex points can be real (i.e., have real coordinates).

[7] Show that $zy^2 + 6z^2y - x^3 - zx^2 + 6z^2x = 0$ is a smooth cubic curve, and find its three real flex points. (Hint: draw a careful graph.)

[8] Let $C = V(F)$ be a smooth curve defined by some irreducible $F \in \mathbf{C}[x, y, z]$ of degree $d > 1$. Determine the degree of the dual curve $C^{\vee}$ in terms of $d$. (Hint: Intersect $C^{\vee}$ with a line in $(\mathbf{P}^2)^{\vee}$ and use Bézout's Theorem.)