M902-2009 Assignment 8: Due Friday April 17

Instructions: Do any three of the following problems.

For Problem 1, use the following definition: Let $k$ be a field. Given $0 \neq f \in k[x_1, \ldots, x_n]$, let $f_t$ be the sum of the terms of $f$ of degree $t$ (and we set $f_t = 0$ if $f$ has no terms of degree $t$); thus $f = \sum_t f_t$. We refer to $f_t$ as the \textit{homogeneous component} of $f$ of degree $t$. An ideal $I \subseteq k[x_1, \ldots, x_n]$ is \textit{homogeneous} if and only if whenever $f \in I$, then $f_t \in I$ for every $t$.

(1) If $I \subseteq k[x_1, \ldots, x_n]$ is homogeneous, show that $\sqrt{I}$ is also.

Problem 2 is true for a polynomial ring in any number of variables; the basic idea of the proof is the same, just more involved. Since we used Problem 2 to characterize primary monomial ideals, do not use facts about primary decompositions of monomial ideals to prove Problem 2; give a direct proof. You may use the fact that a monomial ideal is homogeneous.

(2) Let $k$ be a field and let $I \subseteq k[x, y]$ be a monomial ideal. Show that $\sqrt{I}$ is also monomial.

(3) (a) Let $I_1, \ldots, I_r \subseteq A$ be ideals in a commutative ring $A$ with $1 \neq 0$. Show that $\sqrt{\cap_j I_j} = \cap_j \sqrt{I_j}$.

(b) Show that this can fail for infinite intersections.

(4) Let $A$ be a commutative ring (with $1 \neq 0$) and let $P$ be a prime ideal in $A$. Show that $PA[x]$ is a prime ideal in the ring $A[x]$ obtained by adjoining an indeterminate.

(5) Let $A$ be a commutative ring (with $1 \neq 0$) and let $Q$ be a primary ideal in $A$. Show that $QA[x]$ is a primary ideal in the ring $A[x]$ obtained by adjoining an indeterminate.

(6) Let $A = k[x_1, \ldots, x_n]$ be a polynomial ring over a field.

(a) Let $I = (m_1, \ldots, m_r, ab)$ where each $m_i$ is a monomial and $a$ and $b$ are relatively prime monomials. Show that $I = (m_1, \ldots, m_r, a) \cap (m_1, \ldots, m_r, b)$.

(b) If $I$ and $J_1$ and $J_2$ are monomial ideals in $A$, show that $I \cap (J_1 + J_2) = I \cap J_1 + I \cap J_2$. You may assume that intersections and sums of monomial ideals are monomial, and if $S$ is a set of monomials and $J = (S)$ is the ideal generated by $S$, then a monomial $m$ is in $J$ if and only if some element of $S$ divides $m$.

(c) Show $I \cap (J_1 + J_2) = I \cap J_1 + I \cap J_2$ fails in general.

(d) Show how to use (a) and (b) to find a reduced primary decomposition of $(xz, x^2y^2) \subset k[x, y, z]$. (In fact, (a) and (b) suffice to find primary decompositions of monomial ideals in general.)