

## M902-2009 Assignment 8: Due Friday April 17

*Instructions:* Do any three of the following problems.

For Problem 1, use the following definition: Let  $k$  be a field. Given  $0 \neq f \in k[x_1, \dots, x_n]$ , let  $f_t$  be the sum of the terms of  $f$  of degree  $t$  (and we set  $f_t = 0$  if  $f$  has no terms of degree  $t$ ); thus  $f = \sum_t f_t$ . We refer to  $f_t$  as the *homogeneous component* of  $f$  of degree  $t$ . An ideal  $I \subseteq k[x_1, \dots, x_n]$  is *homogeneous* if and only if whenever  $f \in I$ , then  $f_t \in I$  for every  $t$ .

- (1) If  $I \subseteq k[x_1, \dots, x_n]$  is homogeneous, show that  $\sqrt{I}$  is also.

Problem 2 is true for a polynomial ring in any number of variables; the basic idea of the proof is the same, just more involved. Since we used Problem 2 to characterize primary monomial ideals, do not use facts about primary decompositions of monomial ideals to prove Problem 2; give a direct proof. You may use the fact that a monomial ideal is homogeneous.

- (2) Let  $k$  be a field and let  $I \subseteq k[x, y]$  be a monomial ideal. Show that  $\sqrt{I}$  is also monomial.
- (3) (a) Let  $I_1, \dots, I_r \subseteq A$  be ideals in a commutative ring  $A$  with  $1 \neq 0$ . Show that  $\sqrt{\bigcap_j I_j} = \bigcap_j \sqrt{I_j}$ .  
(b) Show that this can fail for infinite intersections.
- (4) Let  $A$  be a commutative ring (with  $1 \neq 0$ ) and let  $P$  be a prime ideal in  $A$ . Show that  $PA[x]$  is a prime ideal in the ring  $A[x]$  obtained by adjoining an indeterminate.
- (5) Let  $A$  be a commutative ring (with  $1 \neq 0$ ) and let  $Q$  be a primary ideal in  $A$ . Show that  $QA[x]$  is a primary ideal in the ring  $A[x]$  obtained by adjoining an indeterminate.
- (6) Let  $A = k[x_1, \dots, x_n]$  be a polynomial ring over a field.
- (a) Let  $I = (m_1, \dots, m_r, ab)$  where each  $m_i$  is a monomial and  $a$  and  $b$  are relatively prime monomials. Show that  $I = (m_1, \dots, m_r, a) \cap (m_1, \dots, m_r, b)$ .
- (b) If  $I$  and  $J_1$  and  $J_2$  are monomial ideals in  $A$ , show that  $I \cap (J_1 + J_2) = I \cap J_1 + I \cap J_2$ . You may assume that intersections and sums of monomial ideals are monomial, and if  $S$  is a set of monomials and  $J = (S)$  is the ideal generated by  $S$ , then a monomial  $m$  is in  $J$  if and only if some element of  $S$  divides  $m$ .
- (c) Show  $I \cap (J_1 + J_2) = I \cap J_1 + I \cap J_2$  fails in general.
- (d) Show how to use (a) and (b) to find a reduced primary decomposition of  $(xz, x^2y^2) \subset k[x, y, z]$ . (In fact, (a) and (b) suffice to find primary decompositions of monomial ideals in general.)