**M902-2009 Assignment 6: Due Wednesday March 11**

*Instructions*: Do any three of the following problems.

(1) Show that $\mathbf{Q}$ is a flat $\mathbf{Z}$-module but not faithfully flat and not projective.

(2) Give an example of a non-Noetherian module $M$ over a commutative ring $A$ such that $M_P$ is a Noetherian $A$-module for every prime $P \subset A$ (hence being Noetherian is not a local property.)

(3) Let $m$ and $n$ be two positive integers with greatest common divisor $d$. Show that $\mathbf{Z}/m\mathbf{Z} \otimes_{\mathbf{Z}} \mathbf{Z}/n\mathbf{Z} \cong \mathbf{Z}/d\mathbf{Z}$.

(4) Let $m$ and $n$ be two positive integers with greatest common divisor $d$. Show that $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Z}/m\mathbf{Z}, \mathbf{Z}/n\mathbf{Z}) \cong \mathbf{Z}/d\mathbf{Z}$.

(5) Let $A$ be a commutative ring, $f \in A$. Let $\Phi : A[x] \to A_f$ be defined by $ax^n \mapsto a/f^n$. Show that $\Phi$ is surjective with kernel $(xf - 1)$. Conclude $A_f \cong A[x]/(xf - 1)$.

(6) Let $A$ be a commutative ring. We say $A$ is a *Hilbert ring* if, given any ideal $I \subsetneq A$, the radical $\sqrt{I}$ of $I$ is the intersection of the maximal ideals that contain $I$. One way to state Hilbert's version of the Nullstellensatz is that the complex polynomial ring $\mathbf{C}[x_1, \ldots, x_n]$ is a Hilbert ring. In fact, show that $k[x_1, \ldots, x_n]$ is a Hilbert ring for any field $k$.