M902-2009 Assignment 5: Due Wednesday February 25

Instructions: Do any three of the following problems.

(1) Say $K \to M \to N \to 0$ is an exact sequence of left A-modules, where A is a ring. If N and K are finitely generated, show that M is finitely generated.

(2) Say $0 \to K \to M \to N \to 0$ is an exact sequence of left A-modules, where A is a ring. If N is finitely presented and M is finitely generated, show that K is finitely generated. (Hint: apply the snake lemma.)

(3) Let A be a finite product of fields. Show that every A-module M is locally free.

(4) Let A be a finite product of fields. Show that every A-module M is projective.

(5) Let A be a Boolean ring, let I be a maximal ideal and let $S = A \setminus I$. Show that $S_I^{-1}A = A_I$ is a field, isomorphic to A/I.

(6) Let A be a ring and let M, N, P and Q be left A-modules. We discussed in class that $\mathrm{Hom}_A(Q, \cdot)$ is a left exact covariant functor and that $\mathrm{Hom}_A(\cdot, Q)$ is a left exact contravariant functor; i.e., if $M \to N \to P \to 0$ is an exact sequence, then so is $0 \to \mathrm{Hom}_A(P,Q) \to \mathrm{Hom}_A(N,Q) \to \mathrm{Hom}_A(M,Q)$, and if $0 \to M \to N \to P$ is an exact sequence, then so is $0 \to \mathrm{Hom}_A(Q,M) \to \mathrm{Hom}_A(Q,N) \to \mathrm{Hom}_A(Q,P)$. Weak converses also hold. Pick and prove either one of the following.

  (a) If $0 \to \mathrm{Hom}_A(P,Q) \to \mathrm{Hom}_A(N,Q) \to \mathrm{Hom}_A(M,Q)$ is exact for all A-modules Q, then $M \to N \to P \to 0$ is exact.

  (b) If $0 \to \mathrm{Hom}_A(Q,M) \to \mathrm{Hom}_A(Q,N) \to \mathrm{Hom}_A(Q,P)$ is exact for all A-modules Q, then $0 \to M \to N \to P$ is exact.