M902-2009 Assignment 4: Due Wednesday February 18

Instructions: Do any three of the following problems.

(1) Do either (a) or (b). (In fact, there is an example that does both!)
   (a) Give an example of a functor F from **Z**-modules to **Z**-modules which preserves injectivity but is not left exact.
   (b) Give an example of a functor F from **Z**-modules to **Z**-modules which preserves surjectivity but is not right exact.

(2) Let R be a ring and F a functor from R-modules to R-modules. Show that if F preserves exactness on short exact sequences, then F is exact for every sequence.

(3) Give an example of a ring R and an R-module M such that the functor $F = \mathrm{Hom}_R(M, \cdot\,)$ is not right exact. Justify your example.

(4) Let $S \subseteq A$ be a multiplicative subset of a commutative ring A. Show that the functor F defined on A-modules by $F(M) = S^{-1}M$ is exact.

(5) Let A be a commutative ring A. Consider a sequence $M \overset{f}{\to} N \overset{g}{\to} P$ of A-modules. Given a prime ideal $Q \subset A$ and an A-module L, let L_Q denote $S_Q^{-1}L$, where $S_Q = A \setminus Q$. Given an element $s \in A$, let L_s denote $S_s^{-1}L$ where $S_s = \{1, s, s^2, \dots\}$. Show that the following are equivalent (where f_{s_i} and g_{s_i} denote the maps induced by f and g, and f_Q and g_Q denote the maps induced by f and g):
   (a) $M \overset{f}{\to} N \overset{g}{\to} P$ is exact;
   (b) there are elements $s_1, \dots, s_r$ generating (1) such that $M_{s_i} \overset{f_{s_i}}{\to} N_{s_i} \overset{g_{s_i}}{\to} P_{s_i}$ is exact for each i;
   (c) $M_Q \overset{f_Q}{\to} N_Q \overset{g_Q}{\to} P_Q$ is exact for every prime ideal $Q \subset A$.
   (What this problem says is that "exactness is local" because it is enough that it be checked locally.)

(6) If $M_1, \dots, M_r \subset A$ are distinct maximal ideals of a commutative ring A, show that $\Pi_{i=1}^r M_i = \cap_{i=1}^r M_i$.