M902-2009 Assignment 4: Due Wednesday February 18

Instructions: Do any three of the following problems.

(1) Do either (a) or (b). (In fact, there is an example that does both!)
   (a) Give an example of a functor $F$ from $\mathbb{Z}$-modules to $\mathbb{Z}$-modules which preserves injectivity but is not left exact.
   (b) Give an example of a functor $F$ from $\mathbb{Z}$-modules to $\mathbb{Z}$-modules which preserves surjectivity but is not right exact.

(2) Let $R$ be a ring and $F$ a functor from $R$-modules to $R$-modules. Show that if $F$ preserves exactness on short exact sequences, then $F$ is exact for every sequence.

(3) Give an example of a ring $R$ and an $R$-module $M$ such that the functor $F = \text{Hom}_R(M, \cdot)$ is not right exact. Justify your example.

(4) Let $S \subseteq A$ be a multiplicative subset of a commutative ring $A$. Show that the functor $F$ defined on $A$-modules by $F(M) = S^{-1}M$ is exact.

(5) Let $A$ be a commutative ring $A$. Consider a sequence $M \xrightarrow{f} N \xrightarrow{g} P$ of $A$-modules. Given a prime ideal $Q \subset A$ and an $A$-module $L$, let $L_Q$ denote $S_Q^{-1}L$, where $S_Q = A \setminus Q$. Given an element $s \in A$, let $L_s$ denote $S_s^{-1}L$ where $S_s = \{1, s, s^2, \ldots \}$. Show that the following are equivalent (where $f_s$, $g_s$, $f_Q$, and $g_Q$ denote the maps induced by $f$ and $g$, and $f_Q$ and $g_Q$ denote the maps induced by $f$ and $g$):
   (a) $M \xrightarrow{f} N \xrightarrow{g} P$ is exact;
   (b) there are elements $s_1, \ldots, s_r$ generating (1) such that $M_{s_i} \xrightarrow{f_{s_i}} N_{s_i} \xrightarrow{g_{s_i}} P_{s_i}$ is exact for each $i$;
   (c) $M_Q \xrightarrow{f_Q} N_Q \xrightarrow{g_Q} P_Q$ is exact for every prime ideal $Q \subset A$.
(What this problem says is that “exactness is local” because it is enough that it be checked locally.)

(6) If $M_1, \ldots, M_r \subset A$ are distinct maximal ideals of a commutative ring $A$, show that $\prod_{i=1}^r M_i = \cap_{i=1}^r M_i$. 