M902-2009 Assignment 3: Due Wednesday February 11

Instructions: Do any three of the following problems.

(1) Give an example of a ring R and an element $a \in R$ such that a has neither a left nor right inverse, but $RaR = R$.

(2) Let R be a ring and $P \subsetneq R$ an ideal. Show that P is prime if and only if either $a \in P$ or $b \in P$ whenever $a, b \in R$ are elements such that $arb \in P$ for all $r \in R$.

(3) Let R be a ring. Show that any maximal ideal $M \subset R$ is prime.

(4) Give an example of a ring R and a prime ideal $P \subset R$ such that there are elements $a, b \in R$ with $ab \in P$ but neither $a \in P$ nor $b \in P$. (Hint: look at matrices.)

(5) Give an example of a maximal ideal M in a ring R such that R/M is not a division ring.

(6) Let R be a ring and let S be the set of all primes ideals of R. Show that S has elements that are minimal with respect to inclusion. (In combination with Problem 3, we thus see in every ring that the set of primes has both maximal and minimal elements.)