M902-2009 Assignment 2: Due Monday February 2

Instructions: Do any three of the following problems.

(1) Let $k \subseteq E \subseteq K$ be fields with $K$ finitely generated over $k$. Prove that $E$ also is finitely generated over $k$. (Given fields $k \subseteq E \subseteq K = k(T)$ where $T$ is a finite set algebraically independent over $k$, one version of Hilbert’s 14th problem is whether the ring $E \cap k[T]$ must be finitely generated over $k$. This is a much harder problem than showing the field $E$ is finitely generated. Non-finitely generated examples of rings $E \cap k[T]$ were found in the late 50s by Nagata, based partly on previous work by Zariski and others.)

(2) Let $k$ be a field and let $x$ and $t$ be indeterminates. Let $y \in k[x]$ have degree $n \geq 1$. Let $g(t)$ be the image of $y$ under the $k$-homomorphism sending $x$ to $t$.

(a) Prove $y$ is transcendental over $k$.

(b) Prove $g - y$ is irreducible in $k(y)[t]$.

(c) Prove that $[k(x) : k(y)] = n$.

(3) Let $R$ be any ring. Define new operations $\oplus$ and $\otimes$ as follows: $a \oplus b = a + b + 1_R$ and $a \otimes b = ab + a + b$. You may assume the fact that $S = (R, \oplus, \otimes)$ is a ring. Explicitly determine the additive and multiplicative identities of $S$, the additive inverse of an element, and show that $S$ is isomorphic to $R$.

(4) A ring $R$ is said to be Boolean if $a^2 = a$ for each $a \in R$. Prove that a Boolean ring is commutative (i.e., multiplication is commutative).

(5) Let $A$ be an abelian group. Let $R$ be the set of group homomorphisms $A \rightarrow A$. If $f, g \in M$, define $f + g$ to be the map $(f + g)(a) = f(a) + g(a)$ for each $a \in A$. Define $fg$ to be the function $(fg)(a) = f(g(a))$. You may assume the fact that $R$ is a ring with respect to these operations.

(a) Show that the subset $B$ of elements of $R$ with multiplicative inverses is a group isomorphic to the group $G$ of automorphisms of $A$.

(b) If $k$ is a field and $A$ is a $k$-vector space of dimension $n$, show that $R$ is isomorphic to the ring $M_n(k)$ of $n \times n$ matrices over $k$. 

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