M902-2009 Assignment 2: Due Monday February 2

Instructions: Do any three of the following problems.

(1) Let $k \subseteq E \subseteq K$ be fields with K finitely generated over k. Prove that E also is finitely generated over k. (Given fields $k \subseteq E \subseteq K = k(T)$ where T is a finite set algebraically independent over k, one version of Hilbert's 14th problem is whether the ring $E \cap k[T]$ must be finitely generated over k. This is a much harder problem than showing the field E is finitely generated. Non-finitely generated examples of rings $E \cap k[T]$ were found in the late 50s by Nagata, based partly on previous work by Zariski and others.)

(2) Let k be a field and let x and t be indeterminates. Let $y \in k[x]$ have degree $n \geq 1$. Let $g(t)$ be the image of y under the k-homomorphism sending x to t.

(a) Prove y is transcendental over k.

(b) Prove $g - y$ is irreducible in $k(y)[t]$.

(c) Prove that $[k(x) : k(y)] = n$.

(3) Let R be any ring. Define new operations $\oplus$ and $\otimes$ as follows: $a \oplus b = a + b + 1_R$ and $a \otimes b = ab + a + b$. You may assume the fact that $S = (R, \oplus, \otimes)$ is a ring. Explicitly determine the additive and multiplicative identities of S, the additive inverse of an element, and show that S is isomorphic to R.

(4) A ring R is said to be Boolean if $a^2 = a$ for each $a \in R$. Prove that a Boolean ring is commutative (i.e., multiplication is commutative).

(5) Let A be an abelian group. Let R be the set of group homomorphisms $A \to A$. If $f, g \in M$, define $f + g$ to be the map $(f + g)(a) = f(a) + g(a)$ for each $a \in A$. Define fg to be the function $(fg)(a) = f(g(a))$. You may assume the fact that R is a ring with respect to these operations.

(a) Show that the subset B of elements of R with multiplicative inverses is a group isomorphic to the group G of automorphisms of A.

(b) If k is a field and A is a k-vector space of dimension n, show that R is isomorphic to the ring $M_n(k)$ of $n \times n$ matrices over k.