M902-2009 Assignment 1: Due Monday January 26

Instructions: Do any three of the following problems.

(1) (From the June 2008 Comprehensive Exam) Let $K = \mathbf{Q}(i, \sqrt{2})$; you may assume this is an extension of $\mathbf{Q}$ of degree 4. For any field F, let F^* denote the multiplicative group of non-zero elements of F.
   (a) Show that there are exactly three intermediate fields E_1, E_2, E_3 strictly between $\mathbf{Q}$ and K, and determine them.
   (b) Let N denote the norm $N_{\mathbf{Q}}^K$. Show that $N(1 + i + \sqrt{2}) = 8$.
   (c) Show that $1 + i + \sqrt{2} \notin E_1^* E_2^* E_3^*$, that is, $1 + i + \sqrt{2}$ cannot be expressed in the form $\alpha_1\alpha_2\alpha_3$, with $\alpha_i \in E_i$ for each i. ((b) is useful here.)
   (d) Prove that $\alpha^2 \in E_1^* E_2^* E_3^*$ for each $\alpha \in K^*$. (Hint: If E_j is the fixed field for some automorphism σ_j, then $\alpha\sigma_j(\alpha) \in E_j$.)

(2) Let k be a field, let n be a positive integer such that either $\text{char}(k) = 0$ or $(n, \text{char}(k)) = 1$, and assume k contains ζ, a primitive nth root of 1. Let $a \in k$ and let α be a root of $x^n - a$ in $\bar{k}$. Show that $[k(\alpha) : k]$ is the least $\delta > 0$ such that $\alpha^\delta \in k$.

(3) Let K/k be an extension of finite fields. You may assume the fact that any finite extension of a finite field is Galois, with Galois group G generated by a power of Frobenius (specifically, $G = (\phi^r)$, where ϕ is the Frobenius homomorphism, $p = \text{char}(k)$ and $|k| = p^r$). Prove that $N = N_k^K$ is surjective.

(4) Here you will give two solutions to parameterizing rational solutions of $x^2 + y^2 = 1$, a number theoretic solution (part (a)) and an algebraic geometric solution (part (b)).
   (a) Show that every rational solution to $x^2 + y^2 = 1$ is of the form $x = (s^2 - t^2)/(s^2 + t^2)$ and $y = 2st/(s^2 + t^2)$ for some rationals s, t. [Hint: $a^2 + b^2 = 1$ if and only if $(a + bi)(a - bi) = 1$. Apply Hilbert's Theorem 90.]
   (b) Note $y = tx - t$ is the equation of a line through the point $(1, 0)$ of slope t. Show that there is a bijection $t \mapsto (a, b)$ between real solutions $(a, b) \neq (1, 0)$ of $x^2 + y^2 = 1$ and real slopes t of lines $y = tx - t$ passing through the point $(1, 0)$, and express this bijection explicitly. Also show that a and b are both rational if and only if t is rational. [Intuition: t parameterizes the nonvertical lines through $(1, 0)$, every nonvertical line through $(1, 0)$ meets the circle $x^2 + y^2 = 1$ in two points, one of which is $(1, 0)$, and every point of the circle but $(1, 0)$ determines such a line. Thus t parameterizes the points of the circle other than $(1, 0)$.]

(5) Give explicit upper and lower bounds in terms of n on the number of roots of 1 in an extension F of $\mathbf{Q}$ of degree n. In particular, show that the number of roots of 1 is finite, and show that your lower bound is sharp (i.e., for each n, show that there is a field F with $[F : \mathbf{Q}] = n$ such that there number is exactly equal to your lower bound).

(6) Let K be a purely transcendental field extension of a field k. Let $a \in K \setminus k$. Show that a is not algebraic over k.