M901, Assignment 7: Due Monday, November 21, 2011

Instructions: Do any three problems.

Background: A field k is said to be *perfect* if either $\mathrm{char}(k) = 0$, or $\mathrm{char}(k) = p > 0$ and every element of k is the pth power of some element of k (i.e., $k^p = k$, where we define $k^p = \{a^p : a \in k\}$).

(1) Let k be a field. If k is either finite or algebraically closed, show that k is perfect.

(2) Let k be a field with $\mathrm{char}(k) = p > 0$. Show that the subset $k^{\frac{1}{p}} = \{a \in k^a : a^p \in k\}$ is a subfield of k^a isomorphic to k but containing k. Give an example where $k \subsetneq k^{\frac{1}{p}}$. (Note that $(k^{\frac{1}{p}})^p = k$, so your example will give $(k^{\frac{1}{p}})^p = k \subsetneq k^{\frac{1}{p}}$; i.e., $k^{\frac{1}{p}}$ will be an example of a non-perfect field.)

(3) Let k be a field, $\mathrm{char}(k) = p > 0$. Show that the following are equivalent:
  (a) k is a perfect field;
  (b) every irreducible polynomial $f(x) \in k[x]$ is separable; and
  (c) every algebraic extension of k is separable.

(4) Let k be a field with $\mathrm{char}(k) = p > 0$ and let a be algebraic over k. Prove that a is separable over k if and only if $k(a) = k(a^{p^n})$ for every integer $n \geq 1$.

(5) Let k be a field with a and b algebraic over k and a separable over k. Prove that $k(a, b)$ is simple over k.

(6) Let k be a field with $\mathrm{char}(k) = p > 0$. Let x and y be indeterminates. Prove that $k(x^p, y^p) \subset k(x, y)$ has infinitely many distinct intermediate fields.