M901, Assignment 5: Due Friday, October 28, 2011

Instructions: Do any three problems.

Background: Let N and H be groups and let $\theta : H \to \mathrm{Aut}(N)$ be a homomorphism. We define $N \rtimes_\theta H$ to be the set $N \times H$ with the composition law $(n_1, h_1)(n_2, h_2) = (n_1(\theta(h_1))(n_2), h_1, h_2)$. It is now easy to check that $(n, h)(1_N, 1_H) = (n((\theta(h))(1_N)), h1_H) = (n1_N, h1_H) = (n, h)$ and similarly that $(1_N, 1_H)(n, h) = (1_N((\theta(1_H))(n)), 1_H h) = (1_N n, 1_H h) = (n, h)$, so $(1_N, 1_H)$ is an identity. Moreover,

$$((n_1, h_1)(n_2, h_2))(n_3, h_3) = (n_1((\theta(h_1))(n_2)), h_1h_2)(n_3, h_3) = (n_1((\theta(h_1))(n_2))((\theta(h_1h_2))(n_3)), h_1h_2h_3)$$

while

$$(n_1, h_1)((n_2, h_2)(n_3, h_3)) = (n_1, h_1)(n_2((\theta(h_2))(n_3)), h_2h_3) = (n_1((\theta(h_1))(n_2((\theta(h_2))(n_3)))), h_1h_2h_3),$$

but these are the same (and hence the composition law is associative) since

$$(\theta(h_1))(n_2((\theta(h_2))(n_3))) = ((\theta(h_1))(n_2))((\theta(h_1))((\theta(h_2))(n_3))) = ((\theta(h_1))(n_2))((\theta(h_1h_2))(n_3)).$$

Note that $(n, h) = (n, 1_H)(1_N, h)$. It is convenient to identify N with $N \times (1_H)$ and H with $(1_N) \times H$; i.e., $n \in N$, regarded as an element of $N \rtimes_\theta H$, is $(n, 1_H)$, and $h \in H$, regarded as an element of $N \rtimes_\theta H$, is $(1_N, h)$. With these conventions nh denotes $(n, 1_H)(1_N, h)$; i.e., $nh = (n, h)$. If we do this, then we should regard hn as denoting $(1_N, h)(n, 1_H) = ((\theta(h))(n), h)$, which we can denote as $(\theta(h)(n))h$. It is convenient to denote $(\theta(h))(n)$ by n^h. Combining the two notations gives $hn = n^h h$. Let's check associativity using these conventions:

$$(n_1h_1)((n_2h_2)(n_3h_3)) = (n_1h_1)((n_2n_3^{h_2})(h_2h_3)) = (n_1(n_2n_3^{h_2})^{h_1})(h_1h_2h_3) = (n_1n_2^{h_1}n_3^{h_1h_2})(h_1h_2h_3)$$

and

$$((n_1h_1)(n_2h_2))(n_3h_3) = ((n_1n_2^{h_1})(h_1h_2))n_3h_3 = (n_1n_2^{h_1}n_3^{h_1h_2})(h_1h_2h_3).$$

Note that under these identifications we have $1_N = 1_H = 1_{N \rtimes_\theta H}$.

(1) (a) Show that (n, h) has an inverse in $N \rtimes_\theta H$ by writing it down explicitly and checking. Conclude that $N \rtimes_\theta H$ is a group.
   (b) Using the conventions above, show that $n^h = hnh^{-1}$. (Thus $\theta(h)$ applied to N in $N \rtimes_\theta H$ is just conjugation by h.)
   (c) Show that N (regarded as $N \times (1_H)$) is a normal subgroup of $N \rtimes_\theta H$.

(2) Let N and H be groups and let $\theta : H \to \mathrm{Aut}(N)$ be a homomorphism. Show that $N \rtimes_\theta H$ is abelian if and only if θ is trivial and N and H are abelian.

(3) Let N and H be groups and let $\theta : H \to \mathrm{Aut}(N)$ be a homomorphism. Let $\phi : N \times H \to N \rtimes_\theta H$ be the identity on elements. Show that the following are equivalent.
   (a) ϕ is an isomorphism;
   (b) ϕ is a homomorphism; and
   (c) θ is trivial.

(4) Let G be a group. Let $\mathrm{Inn}(G)$ be the group of *inner automorphims*, i.e., the subgroup of $\mathrm{Aut}(G)$ of automorphisms of the form α_g for $g \in G$, defined for any $h \in G$ by $\alpha_g(h) = ghg^{-1}$. Show that $\mathrm{Inn}(G) \lhd \mathrm{Aut}(G)$. (Aside: the quotient $\mathrm{Aut}(G)/\mathrm{Inn}(G)$ is known as the outer automorphism group, sometimes denoted $\mathrm{Out}(G)$, while an automorphism which is not inner is called an outer automorphism. Unfortunately, the elements of $\mathrm{Out}(G)$ are not usually themselves automorphisms, but rather cosets of the group of inner automorphisms.)

(5) Construct a non-abelian semi-direct product $N \rtimes_\theta H$ of order 27, where neither N nor H has order 27.

(6) Let G be an abelian group and let H be a subgroup of G. Let $q : G \to G/H$ be the quotient.
   (a) If there is a homomorphism $s : G/H \to G$ such that the composition qs is the identity, show that $\phi : H \times (G/H) \to G$ defined as $\phi(h, x) = h + s(x)$ is an isomorphism.
   (b) Assume that every element of G has order either 1 or 2. Show that there is an injective homomorphism $s : G/H \to G$ such that the composition qs is the identity. [Hint: Apply Zorn's Lemma.]

   (Aside: In the situation of part (a) we have a short exact sequence

$$0 \to H \to G \to G/H \to 0$$

   of abelian groups and we say that s is a splitting or that the short exact sequence is split when there is a homomorphism s with qs being the identity. Given a split short exact sequence, the middle is isomorphic to the direct product of the ends.)

(7) Let G be a group. Show that $|\mathrm{Aut}(G)| > 1$ if and only if $|G| > 2$.