

M901, Assignment 4: Due Friday, October 7, 2011

Instructions: Do any three problems.

Background: two categories \mathcal{C} and \mathcal{D} are said to be isomorphic if there exist functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ which are mutually inverse to each other (i.e., the compositions either way give the identity functors). If, however, there are merely natural equivalences between FG and the identity functor, and between GF and the identity, we say that the categories are *equivalent*, or that there is an *equivalence* between the categories. (Aside: the concept of equivalence of categories turns out to be more useful than the concept of isomorphism of categories.)

More background: let X be a set and let R be a set of expressions of the form $x_1^{m_1} \cdots x_r^{m_r}$, where $x_i \in X$ and $m_i \in \{-1, 1\}$ for each i . Let B be the free group on the set X (with respect to a map $\phi : X \rightarrow B$). Assume there is a group G and a map $\gamma : X \rightarrow G$ such that the homomorphism $\Gamma : B \rightarrow G$ induced by γ is surjective and its kernel K is the smallest normal subgroup containing $\phi(x_1)^{m_1} \cdots \phi(x_r)^{m_r}$ for each expression $x_1^{m_1} \cdots x_r^{m_r}$ in R . Then we say that G is defined by generators X and relations R , and that (X, R) is a *presentation* for G . If X is finite, we say G is finitely generated. If R is also finite, we say that G is finitely presented.

One more thing: it is known that two vector spaces (over the same field K) are isomorphic (as vector spaces over K) if and only if they have the same dimension (where the dimension of a vector space is the cardinality of a basis). For similar reasons, if B is a free abelian group on a set X and C is free abelian on a set Y , then B and C are isomorphic if and only if there is a bijection $X \rightarrow Y$. You may assume this if you wish for this problem assignment. You may also assume any of the problems below to do any of the others.

- (1) Let X be a set and define a covariant functor F_X from the category \mathcal{A} of abelian groups to the category \mathcal{S} of sets by defining F_X on objects by $F_X(C) = \text{Mor}_{\mathcal{S}}(X, C)$ and on morphisms $h : C \rightarrow D$ by $(F_X(h))(f) = h \circ f$, which we can write as $h_*(f)$. If an abelian group B together with a map $\phi : X \rightarrow B$ is a free abelian group on the set X (i.e., if it satisfies the universal property that for any map $\Psi : X \rightarrow D$ to an abelian group D there is a unique homomorphism $h_\Psi : B \rightarrow D$ such that

$$\begin{array}{ccc} B & \xrightarrow{h_\Psi} & D \\ \phi \swarrow & & \nearrow \Psi \\ & X & \end{array}$$

commutes), prove that B represents the functor F_X . (This is the converse of what we showed in class; we showed that if B represents F_X , then B is a free abelian group on X with respect to an appropriately defined map $X \rightarrow B$.)

- (2) Let \mathcal{C} be the category whose objects are the real vector spaces \mathbf{R}^n , for $n = 0, 1, 2, \dots$ and whose morphisms are linear transformations between these vector spaces. Let \mathcal{V} be the usual category of real finite dimensional vector spaces. Show that \mathcal{C} and \mathcal{V} are equivalent categories.
- (3) Let X be a set with a map $\phi : X \rightarrow B$ to a group B such that (B, ϕ) is a free group on the set X . Let N be the subgroup generated by all commutators of B . Recall that N is a normal subgroup of B ; thus we have the quotient homomorphism $q : B \rightarrow B/N$. Show that $(B/N, q \circ \phi)$ is a free abelian group on X .
- (4) For any set X and any set R of relations, show that there is a group G with presentation (X, R) . Moreover, if H is another group with presentation (X, R) , show that G and H are isomorphic groups.
- (5) Determine the order of the group generated by $X = \{r, f\}$ with relations $R = \{r^3, f^2, rfrf\}$. Justify your answer.
- (6) Let F_i be a free group on the set X_i for $i = 1, 2$, and let N_i be the commutator subgroup of F_i .
 - (a) If there is a bijection $X_1 \rightarrow X_2$ (i.e., if X_1 and X_2 are isomorphic as sets), show that F_1 and F_2 are isomorphic as groups.
 - (b) Prove the converse. (If F_1 and F_2 are isomorphic as groups, show that so are F_1/N_1 and F_2/N_2 . Now apply (3) to conclude that there is a bijection $X_1 \rightarrow X_2$.)