M901, Final Exam: Thursday, December 15, 2011

Instructions: Do any three problems. You may cite without proof facts proved on the homeworks or previous exams, or other problems on this exam (but you are not allowed to cite Problem m while solving Problem m).

Category Theory

(1) Let $\mathcal{C}$ be the category whose objects are sets and whose arrows are inclusions. Given any family $F = \{C_i : i \in I\}$ of sets, let $C_F = \cap_i C_i$. Prove or disprove the statement: C_F together with the morphisms $\{C_F \subseteq C_i : i \in I\}$ is a categorical product in the category $\mathcal{C}$.

Group Actions

(2) If a group G contains a proper subgroup of finite index, show that G contains a proper normal subgroup of finite index.

(3) Let G act on itself by conjugation. Assume that one of the orbits has exactly two elements. Show that G has a nontrivial proper normal subgroup.

Algebraic Extensions of Fields

(4) Let $k \subseteq K$ be fields. If $a, b \in K$ are algebraic over k such that $[k(a) : k]$ and $[k(b) : k]$ are relatively prime, show that $k \subseteq k(a, b)$ has only finitely many intermediate fields.

(5) Let $k \subset F$ be a Galois extension of fields such that $[F : k] = 30$. Let n be a lower bound on the number of intermediate fields E such that E/k is Galois. Find the largest possible lower bound on n. Justify your answer. [Partial credit is available for finding any lower bound; the larger your lower bound the more partial credit you get.]

(6) Let $k \subseteq K$ be fields such that K is algebraically closed. Show that there is a unique algebraically closed field in K containing k and algebraic over k. (I.e., K contains a unique algebraic closure of k.)

Transcendental Extensions of Fields

(7) Let $k \subseteq K$ be fields such that K contains an element $t \in K$ transcendental over k and such that K is algebraically closed (and hence by Problem 6 above contains a unique algebraic closure E of k). Let $a \in E$, $f = \mathrm{Irr}(a, k, x)$ and $g = \mathrm{Irr}(a, k(t), x)$. Let $h \in K[x]$ be monic and divide f in $K[x]$.
  (a) Show that the coefficients of h are algebraic over k.
  (b) Show that $g = f$.
  (c) Conclude that $[k(t)(a) : k(t)] = [k(a) : k]$.