M901, Exam 1: Wednesday, October 12, 2011

Instructions: Do any three problems. You may assume any problem in doing another, even if you don't do the problem you assume, as long as you don't use A to do B and B to do A, or A to do B and B to do C and C to do A, etc.

(1) If $\{X_i : i \in I\}$ is a family of disjoint sets, show that the union $X = \cup_{i \in I} X_i$ is a categorical coproduct in the category of sets with respect to the inclusions $\phi_i : X_i \to X$. (Do this by showing that the appropriate universal property is satisfied. Be sure to state what the universal property is. If you define a map, be sure to justify that the map is well-defined and satisfies the required property.)

(2) Let A_1 and A_2 be disjoint nontrivial but finite abelian groups, and consider the usual injections $i_j : A_j \to A_1 \times A_2$. Recall that the free product $A_1 \circ A_2$ with the canonical injections $c_i : A_i \to A_1 \circ A_2$ as constructed in class, is a categorical coproduct in the category of all groups. Show that the induced homomorphism $A_1 \circ A_2 \to A_1 \times A_2$ is never an isomorphism. Explain why this shows that $A_1 \times A_2$ is not a categorical coproduct in the category of all groups. [Aside: We saw in class that $A_1 \times A_2$ together with the usual injections $A_i \to A_1 \times A_2$ is a categorical coproduct in the category of abelian groups. This shows that nevertheless $A_1 \times A_2$ is not a coproduct in the bigger category of all groups.]

(3) Let $\mathcal{C}$ be a category with at least one object such that any two objects are isomorphic, and every morphism between any two objects is an isomorphism. For any object A of $\mathcal{C}$, show that $\mathrm{Mor}_{\mathcal{C}}(A, A)$ is a group under composition of morphisms and that all the groups that arise this way are isomorphic to each other (i.e., if A, B are objects of $\mathcal{C}$, then $\mathrm{Mor}_{\mathcal{C}}(A, A)$ and $\mathrm{Mor}_{\mathcal{C}}(B, B)$ are isomorphic groups).

(4) Let F be a representable functor from some category $\mathcal{C}$ to the category $\mathcal{S}$ of sets.
   (a) If F is covariant and $f : A \to B$ is a monomorphism in $\mathcal{C}$, show that $F(f)$ is injective.
   (b) If F is contravariant and $f : A \to B$ is a monomorphism in $\mathcal{C}$, must $F(f)$ be injective? Give a proof that it is or an example showing it need not be.

(5) For each set A in the category of sets, choose a free group

$$(F_A, f_A : A \to F_A)$$

on the set A. Define a functor F from the category of sets to the category of groups by declaring for any set X that $F(X) = F_X$ and for any map $g : X \to Y$ of sets that $F(g)$ is the unique homomorphism $h_{f_Y \circ g} : F_X \to F_Y$ such that $h_{f_Y \circ g} \circ f_X = f_Y \circ g$. Show that F is indeed a functor (a covariant functor), and use general properties of functors (rather than specific constructions of free groups) to show that if g is injective, then so is $F(g)$.

(6) Let $\mathcal{C}$ be a category with at least one object such that any two objects are isomorphic, and every morphism between any two objects is an isomorphism. Given any object A of $\mathcal{C}$, we can form a category $\mathcal{A}$ whose only object is A and whose only morphisms are $\mathrm{Mor}_{\mathcal{C}}(A, A)$ (i.e., $\mathrm{Mor}_{\mathcal{A}}(A, A) = \mathrm{Mor}_{\mathcal{C}}(A, A)$). Show that $\mathcal{A}$ and $\mathcal{C}$ are equivalent categories. (If in the course of your solution you define a functor or a natural transformation, be sure to justify that it is a functor or a natural transformation.)