M901-2008 Assignment 6: Due Monday November 3

Instructions: Do any three of the following problems.

(1) Let n be a positive integer. Let A be the additive subgroup of the rationals consisting of all fractions m/n^i where $m \in \mathbf{Z}$ and $i \geq 1$. Consider the indexed set $\{A_i : i \in \{1, 2, 3, \ldots\}\}$ of abelian groups where $A_i = \mathbf{Z}$ for each i. Define $h_i : A_i \to A_{i+1}$ by $h(m) = nm$. Define $f_i : A_i \to A$ by $f(m) = m/n^i$. Prove that A, with respect to the homomorphisms f_i, is the categorical direct limit $\varinjlim A_i$.

(2) Let I be a partially ordered set with a terminal element i_0 (i.e., $j \leq i_0$ for all $j \in I$). Let A_i, $i \in I$, be an indexed family of objects of some category.

(a) Assume the A_i comprise a direct system of objects (i.e., there are appropriate morphisms $f_{ij} : A_i \to A_j$ whenever $i \leq j$). Prove that A_{i_0}, with respect to the homomorphisms $f_{ji_0} : A_j \to A_{i_0}$ coming from the direct system, is the categorical direct limit $\varinjlim A_i$.

(b) This time assume the A_i comprise an inverse system of objects (i.e., there are appropriate morphisms $f_{ij} : A_i \to A_j$ whenever $j \leq i$). Prove that A_{i_0}, with respect to the homomorphisms $f_{i_0 j} : A_{i_0} \to A_j$ coming from the inverse system, is the categorical inverse limit $\varprojlim A_i$.

(3) Let K be a finite field with q elements. Let $f(x) \in K[x]$ be irreducible. Show that f divides $x^{q^n} - x$ in $K[x]$ if and only if $\deg(f)$ divides n.

(4) Let $K = \mathbf{Z}/2\mathbf{Z}$. Prove that $K[x]/(x^3 + x + 1)$ and $K[y]/(y^3 + y^2 + 1)$ are fields, show they are isomorphic and give an explicit isomorphism.

(5) Let F/K be a finite dimensional extension of fields where K is an infinite field and $\alpha, \beta \in F$. Show that the set T of $t \in K$ such that $K(\alpha, \beta) = K(\alpha + t\beta)$ is either empty or infinite.

(6) Let E/k be an algebraic extension of fields, with $\alpha, \beta \in E$, where α is separable over k.

(a) If $\mathrm{char}(k) = p > 0$, prove that $k(\alpha) = k(\alpha^p)$.

(b) Prove that $k(\alpha, \beta)/k$ is simple.

(7) (a) Let α be algebraic over a field E. If $E(\alpha)/E$ is a normal extension, prove that $\mathrm{Aut}_E(E(\alpha))$ has order $[E(\alpha) : E]_s$.

(b) Let F/E be an extension of fields where $|E| = p^r$ for some prime p, and $F = p^{rs}$. Prove that $\mathrm{Aut}_E(F)$ is cyclic generated by ϕ^r, where $\phi : F \to F$ is Frobenius (i.e., $\phi(c) = c^p$).