M901-2008 Assignment 5: Due Wednesday October 22

Instructions: Do any three of the following problems.

(1) Let $\mathcal{C}$ be a category. Let $\text{Sets}^{\mathcal{C}}$ be the category whose objects are contravariant functors from $\mathcal{C}$ to the category Sets of sets, and whose arrows are natural transformations between functors. Let $F : \mathcal{C} \to \text{Sets}$ be a contravariant functor. Let A and B be objects of $\mathcal{C}$. Show that there is a bijection between $\text{Mor}_{\mathcal{C}}(A, B)$ and $\text{Mor}_{\text{Sets}^{\mathcal{C}}}(h^A, h^B)$, where for any object C of $\mathcal{C}$, h^C denotes the functor $\text{Mor}(\cdot, C)$. (This is a version of Yoneda's Lemma.)

(2) Let X be a topological space. Regard X as a category whose objects are the open subsets, and whose arrows are the inclusion maps of open subsets.

(a) Given two inclusions $U \subseteq W$ and $V \subseteq W$ of open subsets, use the universal property of fiber products to determine the fiber product $U \times_W V$.

(b) Given two inclusions $W \subseteq U$ and $W \subseteq V$ of open subsets, use the universal property of fiber coproducts to determine the fiber coproduct $U \amalg_W V$.

(3) Let m and n be positive integers, let r be the least common multiple and let d be the greatest common divisor.

(a) Consider the quotient homomorphisms $\mathbf{Z} \to \mathbf{Z}/m\mathbf{Z}$ and $\mathbf{Z} \to \mathbf{Z}/n\mathbf{Z}$. Using the appropriate universal property, determine the categorical fiber coproduct $T = \mathbf{Z}/m\mathbf{Z} \amalg_{\mathbf{Z}} \mathbf{Z}/n\mathbf{Z}$ up to isomorphism in the category of abelian groups.

(b) Consider injective homomorphisms $p : \mathbf{Z}/m\mathbf{Z} \to \mathbf{Z}/r\mathbf{Z}$ (where $[1]_m \mapsto [n/d]_r$) and $q : \mathbf{Z}/n\mathbf{Z} \to \mathbf{Z}/r\mathbf{Z}$ (where $[1]_n \mapsto [m/d]_r$). Determine the fiber product $\mathbf{Z}/m\mathbf{Z} \times_{\mathbf{Z}/r\mathbf{Z}} \mathbf{Z}/n\mathbf{Z}$ up to isomorphism in the category of abelian groups.

(4) Let $\mathcal{V}$ be the category of finite dimensional real vector spaces, where the morphisms are vector space isomorphisms. Given an object V in $\mathcal{V}$, let D be the functor from $\mathcal{V}$ to $\mathcal{V}$ where $D(V)$ is the dual space $V^* = \text{Hom}(V, \mathbf{R})$ of linear functionals on V. Given any isomorphism $f : V \to W$ and a linear functional l on W, define $f^*(l)$ to be the linear functional on V defined by $f^*(l)(v) = l(f(v))$. Now for any such f, define $D(f) = (f^*)^{-1}$. Thus D is a covariant functor from $\mathcal{V}$ to $\mathcal{V}$, and so is the double dual, D^2.

(a) Show that any vector space V is canonically isomorphic to its double dual V^{**} (i.e., show that D^2 is naturally equivalent to the identity functor).

(b) Show that D is *not* naturally equivalent to the identity functor.

(5) Two categories $\mathcal{C}$ and $\mathcal{D}$ are said to be isomorphic if there exist functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ which are mutually inverse to each other (i.e., the compositions either way give the identity functors). If, however, there are merely natural equivalences between FG and the identity functor, and between GF and the identity, we say that the categories are equivalent, or that there is an equivalence between the categories. Give an example (with proof) of equivalent but nonisomorphic categories. (The concept of isomorphic categories is not very useful, but the concept of equivalence of categories is.)