M901-2008 Assignment 4: Due Monday September 29

Instructions: Do any three of the following problems.

(1) Consider $G = (\mathbf{Z}/2^n\mathbf{Z})^*$.
  (a) Determine G up to isomorphism for $n = 1$ and $n = 2$.
  (b) For $n \geq 3$, show that G has two elements of order 2, and hence is not cyclic.

(2) The goal of this problem is to determine $\mathrm{Aut}(\mathbf{Z}/2^n\mathbf{Z})$ when $n \geq 3$, using the fact that $\mathrm{Aut}(\mathbf{Z}/2^n\mathbf{Z}) \cong (\mathbf{Z}/2^n\mathbf{Z})^*$. So assume $n \geq 3$.
  (a) Let $i = x2^j$ where x is odd and $j \geq 0$. If $i \geq 3$, show that $i - j \geq 2$, with $i - j \geq 3$ when $i \geq 5$.
  (b) Show 3 has order $n - 2$ in $(\mathbf{Z}/2^n\mathbf{Z})^*$.
  (c) Use (b) and 1(b) to conclude $(\mathbf{Z}/2^n\mathbf{Z})^* \cong (\mathbf{Z}/2^{n-2}\mathbf{Z}) \times (\mathbf{Z}/2\mathbf{Z})$.

(3) Show that a group G of order 150 is solvable. (This is Problem 1 on the January 2007 comprehensive exam.)

(4) Let P be a subgroup of a finite group G.
  (a) Let $\alpha \in \mathrm{Aut}(G)$. Show that $\alpha(N_G(P)) = N_G(\alpha(P))$.
  (b) Let P be a Sylow subgroup of a finite group G. Prove the following are equivalent:
    (i) $P \lhd G$
    (ii) P char G
    (iii) $N_G(P)$ char G

(5) Prove that any nilpotent group is solvable.

(6) For any subgroups H, K of a group G, define (H, K) to be the subgroup generated by the set $\{hkh^{-1}k^{-1} : h \in H, k \in K\}$. Now define $\gamma_1(G) = G$ and for $i \geq 1$ define $\gamma_{i+1}(G) = (\gamma_i(G), G)$. Prove that G is nilpotent if and only if $\gamma_i(G) = (e_G)$ for some i.