M901-2008 Assignment 2: Due Monday September 15

Instructions: Do any three of the following problems.

(1) Prove $F = \langle X | xy = yx \text{ for all } x, y \in X \rangle$ is a free abelian group with respect to the map $f : X \to F$ where f is the composition of the canonical inclusion $i : X \to F(X)$ of X into the free group $F(X)$ with the canonical quotient homomorphism $F(X) \to F$.

(2) Let $f : X \to Y$ be a map of sets. Prove that f is surjective if and only if, whenever h and g are maps $Y \to Z$ of sets with $hf = gf$, then $h = g$.

(3) Let G be a finite group acting on a finite set S.
  (a) Prove that $|G||S| \geq \sum_{s \in S} |Gs|$, with equality if and only if every nonidentity element of G is fixed point free.
  (b) Find a lower bound for $\sum_{s \in S} |Gs|$ which holds if and only if the action is trivial.

(4) Show that any finite group G is finitely presented.

(5) Let $\phi : G_2 \to \mathrm{Aut}(G_1)$ be a homomorphism. Show that $G_1 \rtimes_\phi G_2$ is abelian if and only if ϕ is trivial and G_1 and G_2 are abelian.

(6) Let $p \leq q$ be primes. Prove there exists a nonabelian group of order pq if and only if $p \mid q - 1$.