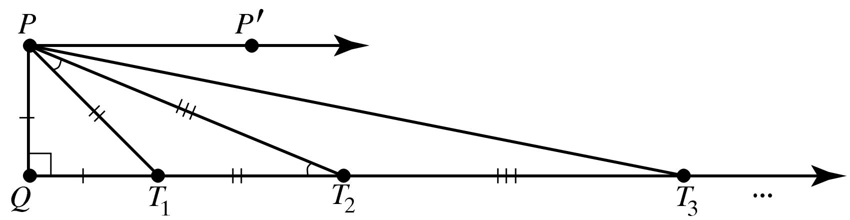
# Angle Sum of Triangles in Hyperbolic Geometry

We will now show that the sum of the angle measures in a triangle on the hyperbolic plane is strictly less than 180°. First we need

**Lemma.** *Suppose $\overline{PQ}$ is a segment and $Q'$ is a point such that $\angle PQQ'$ is a right angle. For every $\epsilon > 0$ there exists a point $T$ on $\overrightarrow{QQ'}$ such that $m(\angle PTQ) < \epsilon°$.*

*Proof.*




Construct a point $P'$ such that $\overleftrightarrow{PP'}$ is perpendicular to $\overleftrightarrow{PQ}$. We choose a sequence of points $T_1, T_2, \ldots$ as follows. First choose $T_1$ such that $\overline{PQ} \cong \overline{QT_1}$. Then choose $T_2$ such that $\overline{PT_1} \cong \overline{T_1T_2}$. Iteratively choose $T_n$ beyond $T_{n-1}$ such that $\overline{PT_{n-1}} \cong \overline{T_{n-1}T_n}$.

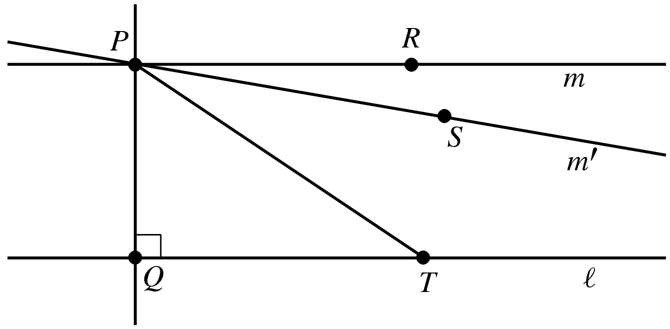
1. Show that $\angle QT_nP \cong \angle T_{n-1}PT_n$.

2. Show that for each $n$, $m(\angle QPT_1) + m(\angle T_1PT_2) + \ldots + m(\angle T_{n-1}PT_n) < 90°$.

3. Finish the proof using contradiction.

**Theorem.** *On the hyperbolic plane, there is a triangle whose angle sum is strictly less than* $180°$.

*Proof.*




Let $\ell$ be a line and $P$ a point off $\ell$. Drop a perpendicular line from $P$ to $\ell$, intersecting at $Q$. Let $m$ be the line through $P$ that is perpendicular to $\overleftrightarrow{PQ}$.

1. Why is $m$ is parallel to $\ell$?

By the Hyperbolic Parallel Postulate (HPP), there is another line $m'$ through $P$ that is parallel to $\ell$. Choose point $S$ on $m'$ to be on the same side of $m$ as $Q$, and point $R$ on $m$ to be on the same side of $\overleftrightarrow{PQ}$ as $S$. Using the Lemma, choose $T$ on $\ell$ so that $m(\angle QTP) < m(\angle SPR)$.

2. Show that $S(\triangle QTP) < 180°$.