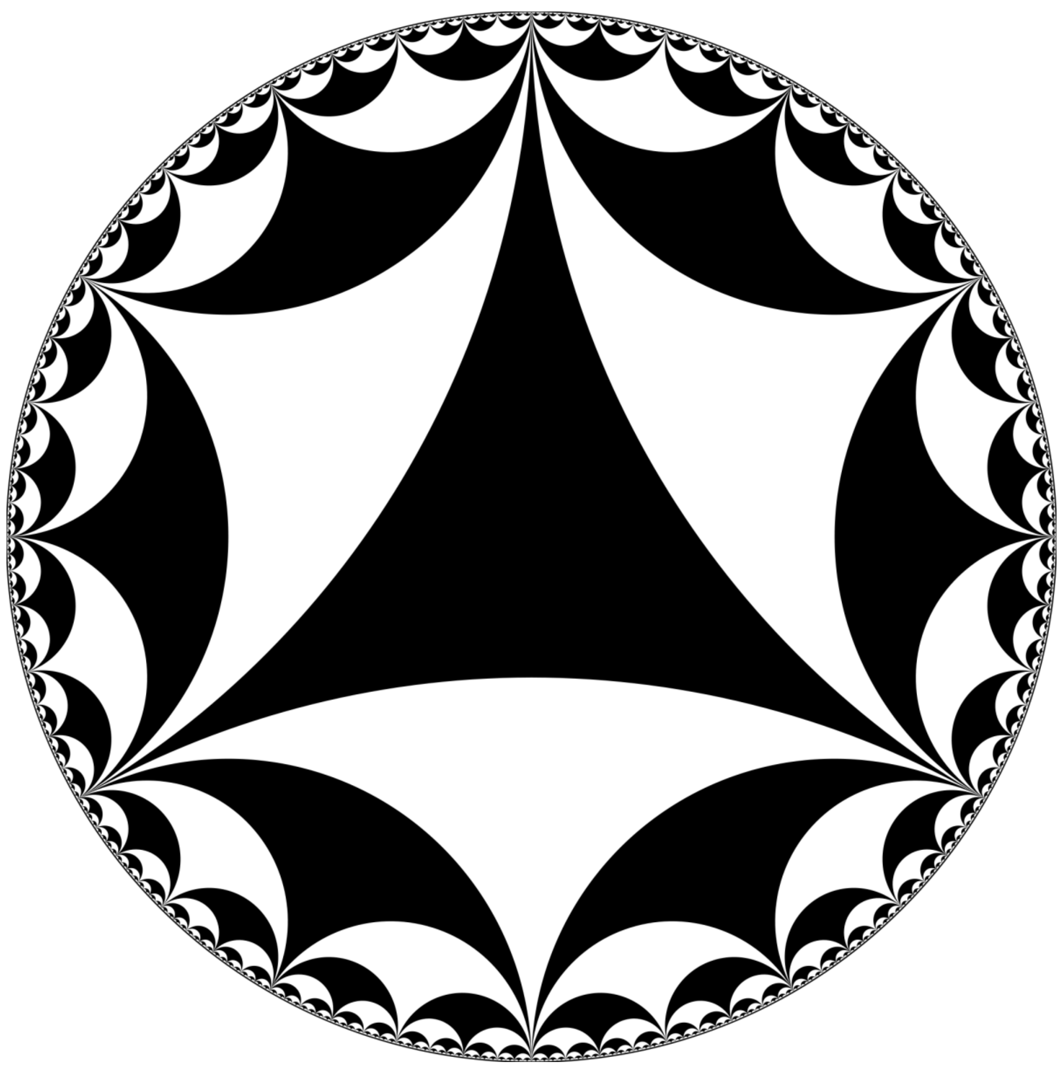
Ideal Triangles and the Area Formula on the Hyperbolic Plane

Following our experimentation, we expect a linear relationship between area and defect, similar to what we had on the sphere between area and excess.

One can rigorously prove such an area formula, but it is a slightly harder road than on the sphere. We will work through some of the more interesting highlights. In this worksheet we will do our work on the Poincaré Disk Model of the hyperbolic plane because it is easier to see the pictures.

First we need to know what an "Ideal Triangle" and a "2/3 Ideal Triangle" are. First off, these are not actually triangles, but rather sets of parallel lines. If you choose 3 points on the boundary of the disk, then there are three lines that go through those points. These three lines make an Ideal Triangle. Since the points on the boundary of the disk are not actually points in the space, this isn't a real triangle, but rather just three pairwise critically parallel lines. The 2/3 Ideal Triangle comes from taking one point in the plane and two points on the boundary. So you have one real angle and the other two "vertices" are on the boundary. One can actually show that if you have two 2/3 Ideal Triangles with the same angle, then they are congruent, and that all Ideal Triangles are congruent.



All triangles pictured are Ideal Triangles.

Fact: One can show that the area on any Ideal Triangle is $\pi\rho^2$, where ρ is the radius of the hyperbolic plane. If α, is the angle of a 2/3 Ideal Triangle then the area of the 2/3 Ideal triangle is $(\pi - \alpha)\rho^2$. Note that as α gets close to 0, the 2/3 Ideal Triangle gets close to an Ideal Triangle. From this information we want to derive the formula for the area of any triangle.

1) Using the Poincare Model, either on the computer or by hand, draw any triangle ΔABC and label the angles as α, β, and γ. Extend rays $\overrightarrow{AB}$, $\overrightarrow{BC}$, and $\overrightarrow{CA}$, This yields three points on the boundary of the circle.

2) Construct the Ideal Triangle through these three points.
3) Use the facts above to derive the formula for the area of ΔABC.

4) Can you rewrite the formula in terms of the defect?

5) Does your formula match what you experimentally found earlier?