**SACCHERI-LEGENDRE THEOREM**

**Theorem** (Saccheri-Legendre Theorem). If one assume Euclid’s postulates other than the parallel postulate, then the sum of the interior angles of a triangle is at most 180°.

**Proof.** **Step 1:** Prove that the angle sum of any two interior angles of a triangle is less than 180°. (For example, given \(\triangle ABC\), to show that \(m(\angle BAC) + m(\angle ACB) < 180°\), let \(E\) be the midpoint of the side opposite \(B\) as in the diagram, and let \(F\) be on the line through \(B\) and \(E\) with \(E\) between \(B\) and \(F\) such that \(BF\) is twice \(BE\). Justify the angle congruences shown in the diagram and conclude that \(m(\angle BAC) + m(\angle ACB) < 180°\).)

![Diagram showing Step 1](image)

**Step 2:** Given any triangle \(\triangle ABC\), prove that there is another triangle having the same angle sum but where one interior angle measures at most \(\frac{m(\angle CAB)}{2}\). (For example, in \(\triangle ABD\) in the diagram below, show that triangles \(\triangle ABC\) and \(\triangle ABD\) have the same interior angle sum but that either \(\angle ADB\) or \(\angle BAD\) has measure at most \(\frac{m(\angle CAB)}{2}\).)

![Diagram showing Step 2](image)

**Step 3:** Suppose a triangle \(\triangle ABC\) has interior angle sum more than 180°, say 180° + \(\epsilon\)° for some \(\epsilon > 0\). By repeatedly using Steps 1 and 2, show that there would then be a triangle, one angle of which has measure less than \(\epsilon\) but nonetheless has the same interior angle sum, and thereby derive a contradiction to \(\triangle ABC\) having interior angle sum 180° + \(\epsilon\)°. \(\square\)

**Question:** How does assuming the parallel postulate allow you to show that the interior angle sum is exactly 180°?