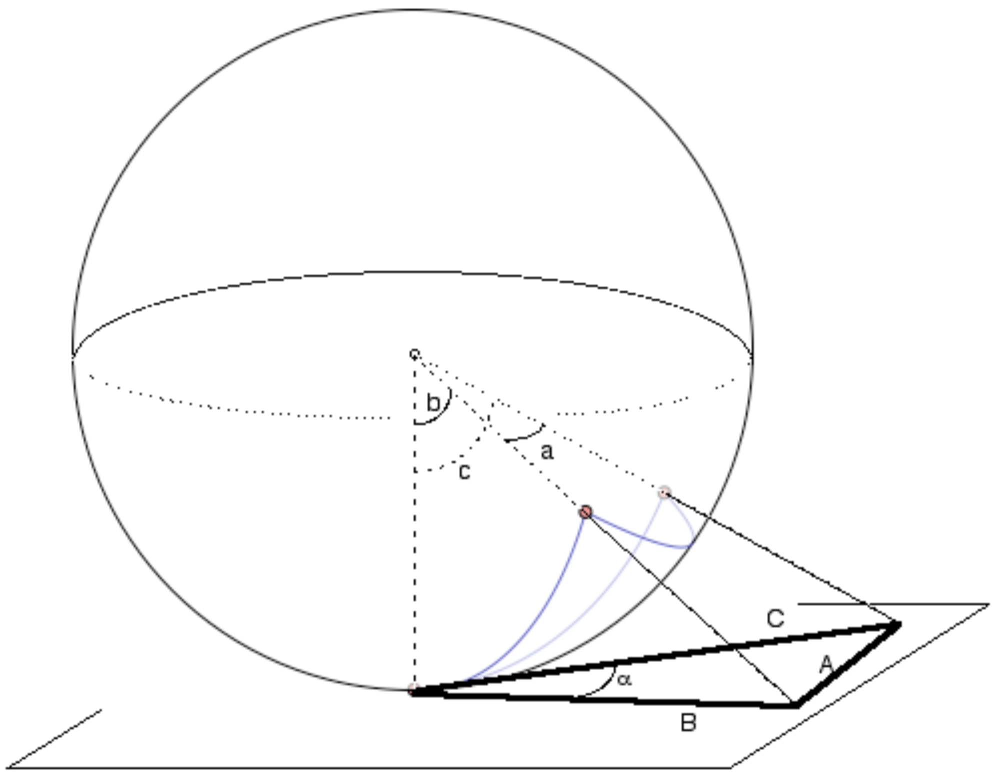
Math 812T: Spherical Pythagorean Theorem & Law of Cosines Activity



We are given a spherical triangle of sides a, b and c (measured in radians) with angle α opposite side a. The plane shown is tangent to the sphere at the vertex of angle α. We can project the spherical triangle onto the tangent plane, where the center of the projection is the center of the sphere. (A projection from the center of a sphere to a tangent plane is called a *gnomonic* projection. Under this projection, great circles on the sphere correspond to lines in the tangent plane.) Assume the sphere has radius 1.

Step 1: Assume α is a right angle. Find the square A^2 of the side whose length is A two ways. Set your results equal and simplify, using the identity $\sec^2(\theta) = 1 + \tan^2(\theta)$. For the first way, use Pythagoras' Theorem in the plane, but express B and C in terms of b and c. For the second way, use the law of cosines on the triangle having side of length A with the opposite vertex being the center of the sphere. Your final result should involve only cosines of a, b and c. The formula you'll get is often referred to as the spherical analogue of Pythagoras' Theorem.

Answer: Pythagoras' Theorem gives $A^2 = B^2 + C^2 = \tan^2(b) + \tan^2(c)$. Let x be the distance from the center of the sphere to the point where the segments of lengths A and B come together, and let y be the distance from the center of the sphere to the point where the segments of lengths A and C come together. Then the law of cosines says $A^2 = x^2 + y^2 - 2xy\cos(a) = \sec^2(b) + \sec^2(c) - 2\sec(b)\sec(c)\cos(a)$. Thus we get

$$\tan^2(b) + \tan^2(c) = \sec^2(b) + \sec^2(c) - 2\sec(b)\sec(c)\cos(a) = 2 + \tan^2(b) + \tan^2(c) - 2\sec(b)\sec(c)\cos(a)$$

which simplifies to $\cos(a) = \cos(b)\cos(c)$.

Step 2: We can make this formula look more like the planar Pythagoras' Theorem. To do so, square it, use the identity $\cos^2(\theta) + \sin^2(\theta) = 1$ to express it in terms of sines rather than cosines, and simplify.

Answer: We get $1 - \sin^2(a) = \cos^2(a) = \cos^2(b)\cos^2(c) = (1 - \sin^2(b))(1 - \sin^2(c))$ which simplifies to

$$\sin^2(a) = \sin^2(b) + \sin^2(c) - \sin^2(b)\sin^2(c).$$

Step 3: Repeat step 1, but don't assume that α is a right angle. Thus you'll have to use the planar law of cosines in place of the planar Pythagorean Theorem. The formula you'll get is the spherical law of cosines, also known as the law of cosines for sides (there is a similar law of cosines for angles).

Answer: The planar law of cosines gives $A^2 = B^2+C^2-2BC\cos(\alpha) = \tan^2(b)+\tan^2(c)-2\tan(b)\tan(c)\cos(\alpha)$. The law of cosines says $A^2 = x^2+y^2-2xy\cos(a) = \sec^2(b)+\sec^2(c)-2\sec(b)\sec(c)\cos(a)$. Thus we get

$$\tan^2(b)+\tan^2(c)-2\tan(b)\tan(c)\cos(\alpha) = \sec^2(b)+\sec^2(c)-2\sec(b)\sec(c)\cos(a)$$

which gives $-2\tan(b)\tan(c)\cos(\alpha) = 2-2\sec(b)\sec(c)\cos(a)$, which simplifies to $\cos(a) = \cos(b)\cos(c)+\sin(b)\sin(c)\cos(\alpha)$.