# Homework 8, due Tuesday, November 27, 2012

Do any 4 of the 6 problems. Each problem is worth 25 points. Solutions will be graded for correctness, clarity and style.

(1) Let X be a topological space and let Y be a set. Let $f : X \to Y$ be a map, not necessarily surjective. Let $\mathcal{T}_Y = \{V \subseteq Y : f^{-1}(V) \text{ is open in } X\}$. Show that $\mathcal{T}_Y$ defines a topology on Y. (In class we used this construction mainly when f is surjective.)

(2) Let X be a topological space and let Y be a set. Let $f : X \to Y$ be a map such that $f(X)$ is a single point $p \in Y$. Let $\mathcal{T}_Y = \{V \subseteq Y : f^{-1}(V) \text{ is open in } X\}$. Show that $\mathcal{T}_Y$ is the discrete topology on Y.

(3) Let σ^k be the standard k-simplex, so $\sigma^k = \{(x_1, \ldots, x_{k+1}) \in \mathbf{R}^{k+1} : x_1 \geq 0, \ldots, x_{k+1} \geq 0, x_1 + \cdots + x_{k+1} = 1\}$. Let τ^k be a k-simplex in $\mathbf{R}^N$, so $\tau^k = \langle v_1, \ldots, v_{k+1} \rangle$ where the points $v_i \in \mathbf{R}^N$ are geometrically independent. Define a map $f : \sigma^k \to \tau^k$ by $f((x_1, \ldots, x_{k+1})) = \sum_i x_i v_i$.
    (a) Show that f is bijective.
    (b) Assuming that f is continuous, show that f is a homeomorphism. This shows that all k-simplices are homeomorphic to the standard k-simplex and hence to each other. [Aside: $h : \mathbf{R}^{k+1} \to \mathbf{R}^N$ defined by $f((a_1, \ldots, a_{k+1})) = \sum_i a_i v_i$ is a linear transformation. It is known and not hard to show that linear transformations are continuous under the standard topologies, basically because a linear transformation just involves addition and multiplication of the variables by constants, and these are continuous. Since f is the restriction of h to σ^k, this means that f is indeed continuous.]

(4) Let $0 \leq r \leq k \leq K$ where r, k and K are integers. Give a formula for the number of k-simplices contained in a K-simplex which contain a given r-simplex. [For example, if $r = 0$, $k = 1$ and $K = 3$, this is asking how many edges of a tetrahedron contain a given vertex of the tetrahedron, so your formula should give 3.]

(5) Consider a 14-gon. Go around the circumference of the 14-gon, labeling the edges in turn as follows: a, b, a^{-1}, c, b^{-1}, c^{-1}, d, e, g, e^{-1}, f, g^{-1}, f^{-1}, d^{-1}. This gives a planar diagram, with labels and exponents, for a multi-holed torus as in the diagram below at left. (The planar diagram can also be shown with labels and arrows as in the diagram below at right. Think of a as specifying an arrow on the corresponding edge of the 14-gon pointing in the direction in which you're going around the 14-gon. Think of a^{-1} as specifying an arrow on the corresponding edge of the 14-gon but pointing in the direction opposite to which you're going around the 14-gon.) Determine the number of holes of the torus you get by making the identifications specified by the labels.
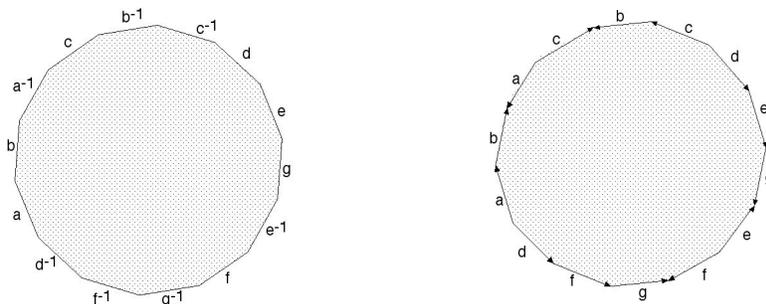
(6) Determine whether the points $(1, 1, 1), (2, 3, 4), (3, 5, 7) \in \mathbf{R}^3$ are geometrically independent.