Homework 2, due Thursday, September 6, 2012

Let $f : X \to Y$ be a mapping of sets. We say f is one to one (or injective) if whenever x_1 and x_2 are in X with $f(x_1) = f(x_2)$ then $x_1 = x_2$. Another way to say it is: f is injective if $x_1 \neq x_2$ always implies $f(x_1) \neq f(x_2)$.

We say f is onto (or surjective) if for each $y \in Y$ there is an $x \in X$ with $f(x) = y$. Another way to say it is: f is surjective if $f(X) = Y$.

And we say f is bijective if f is injective and surjective.

Remember: the goal in writing proofs is not only to be right, but to be understood. Never turn in an assignment without proofreading it. (As Truman Capote said in reference to the Beat writers, "That isn't writing at all, it's typing.") And when you proofread your proofs, check not only for correctness but ask yourself: is it clear? Is it unambiguous? Could I say the same thing more simply? Could I shorten it and still be clear? Would a few extra words be helpful? Also remember that proof writing is a form of writing and thus requires sentences. Proofs with no words are hard to read. A specific general rule is never to begin a sentence with a symbol. So every proof should have at least one sentence and thus at least one word!

Each problem is worth 20 points. Solutions will be graded for correctness, clarity and style.

(1) Let X be a set with $|X| > 1$. Show that the indiscrete topology is not a metric topology on X. (I.e., show that if d is a metric on X, then some union of open discs is not an open set in the indiscrete topology.)

Solution: Let d be any metric on X. Since $|X| > 1$, we can pick $x, y \in X$ with $x \neq y$. Let $U = D_X(x, r) = \{z \in X : d(x, z) < r\}$, where $r = d(x, y)$. Thus U is an open set in the topology induced by d on X, but $y \notin U$ since $d(x, y) = r$, so $U \neq X$, and $x \in U$ since $r > 0$, so $U \neq \varnothing$. Thus U is not an open set in the indiscrete topology (by definition, the only open sets in the indiscrete topology are $\varnothing$ and X). Thus the indiscrete topology is not a metric topology.

(2) Let $f : X \to Y$ be a map of sets. Prove that $f : X \to Y$ is injective if and only if $f(A) \cap f(B) \subseteq f(A \cap B)$ for all subsets $A, B \subseteq X$.

Solution: We need to prove if $f : X \to Y$ is injective, then $f(A) \cap f(B) \subseteq f(A \cap B)$ for all subsets $A, B \subseteq X$, and we need to prove that if $f(A) \cap f(B) \subseteq f(A \cap B)$ for all subsets $A, B \subseteq X$, then f is injective.

We first prove the first implication. Assume $f : X \to Y$ is injective. Let $y \in f(A) \cap f(B)$. Thus $y \in f(A)$, so $y = f(a)$ for some $a \in A$, and $y \in f(B)$, so $y = f(b)$ for some $b \in B$. But f is injective, so $f(a) = f(b)$ implies $a = b$, hence $a = b \in A \cap B$, so $y = f(a) = f(b) \in f(A \cap B)$. Thus $f(A) \cap f(B) \subseteq f(A \cap B)$.

Now we'll prove the second implication. Assume $f(A) \cap f(B) \subseteq f(A \cap B)$ for all subsets $A, B \subseteq X$. Let $x_1, x_2 \in X$. Let $A = \{x_1\}$ and $B = \{x_2\}$. If $x_1 \neq x_2$, then $A \cap B = \varnothing$, so $f(A) \cap f(B) \subseteq f(A \cap B) = f(\varnothing) = \varnothing$; i.e., $f(A) \cap f(B) = \varnothing$, so $f(x_1) \neq f(x_2)$, so f is injective.

(3) Let $f : X \to Y$ be a map of sets. Show that f is surjective if and only if $C \subseteq f(f^{-1}(C))$ for all subsets $C \subseteq Y$.

Solution: Assume f is surjective. Let $c \in C$. Since f is surjective, there is an $x \in X$ with $f(x) = c$, hence $x \in f^{-1}(C)$ so $c = f(x) \in f(f^{-1}(C))$. Thus $C \subseteq f(f^{-1}(C))$.

Now assume $C \subseteq f(f^{-1}(C))$ for all subsets $C \subseteq Y$. Let $y \in Y$ and consider $C = \{y\}$. Since $\{y\} = C \subseteq f(f^{-1}(C))$, we see $y \in f(f^{-1}(C))$, so $y = f(x)$ for some $x \in f^{-1}(C) \subseteq X$. Thus each $y \in Y$ is $f(x)$ for some $x \in X$, so f is surjective.

(4) Let X be the set of all continuous functions $f : [0, 1] \to \mathbf{R}$. Given $f, g \in X$, define $d(f, g) = \int_0^1 |f(t) - g(t)|\, dt$. Prove that d is a metric on X. (You may assume usual facts from calculus.)

Solution: We first show that $d(f, g) \geq 0$ for all f and g, with $d(f, g) > 0$ if and only if $f \neq g$. Since $|f(t)-g(t)| \geq 0$ for all $t \in [0, 1]$, we know $d(f, g) = \int_0^1 |f(t)-g(t)|\, dt \geq 0$. If $f = g$, then $d(f, g) = \int_0^1 |f(t) - g(t)|\, dt = \int_0^1 0\, dt = 0$, while if $f \neq g$, then $f(t) \neq g(t)$ for some $t \in [0, 1]$, so $|f(t) - g(t)| > 0$ for some $t \in [0, 1]$, hence there is a positive amount of area under $|f(t) - g(t)|$, hence $d(f, g) = \int_0^1 |f(t) - g(t)|\, dt > 0$. Since $|f(t) - g(t)| = |g(t) - f(t)|$, we have $d(f, g) = d(g, f)$. Finally, let $h \in X$. For any $t \in [0, 1]$, let $a = f(t)-g(t)$, $b = g(t)-h(t)$, and let $c = f(t)-h(t)$. If a and b are both nonnegative or both nonpositive, then $|a| + |b| = |a + b|$, while if a and b have different signs, then $|a + b| < |a| + |b|$. Thus we always have $|a + b| \leq |a| + |b|$, hence $|f(t)-h(t)| = |f(t)-g(t)+g(t)-h(t)| = |a+b| \leq |a|+|b| = |f(t)-g(t)|+|g(t)-h(t)|$, so $d(f, g)+d(g, h) = \int_0^1 |f(t) - g(t)|\, dt + \int_0^1 |g(t) - h(t)|\, dt = \int_0^1 (|f(t) - g(t)| + |g(t) - h(t)|)\, dt \geq \int_0^1 |f(t) - h(t)|\, dt$. Thus d is a metric.

(5) Let X be a nonempty set and let $d : X \times X \to \mathbf{R}$ be a function such that $d(x, y) = 0$ if and only if $x = y$ and such that $d(x, y) + d(z, y) \geq d(x, z)$ for all $x, y, z \in X$. Show that $d(x, y) \geq 0$ and $d(x, y) = d(y, x)$ for all $x, y \in X$. Conclude that d is a metric.

Solution: By hypothesis we have $d(z, x) = d(x, x) + d(z, x) \geq d(x, z)$ (take $y = x$) for all $x, z \in X$. Since this is true for all $x, z \in X$ we can switch x and z and get $d(x, z) \geq d(z, x)$. Thus $d(x, z) = d(z, x)$ for all $x, z \in X$.

Now let $z = x$. We get $2d(x, y) = d(x, y) + d(x, y) \geq d(x, x) = 0$ for all $x, y \in X$, so $d(x, y) \geq 0$ for all $x, y \in X$. We know by hypothesis that $d(x, y) = 0$ if and only if $x = y$, so we have $d(x, y) \geq 0$ for all $x, y \in X$, with $d(x, y) = 0$ iif and only if $x = y$. And we know $d(x, y) + d(y, z) = d(x, y) + d(z, y) \geq d(x, z)$ for all $x, y, z \in X$. Thus d is a metric on X.