Exam 2, Thursday, November 6, 2012

Do any 8 of the 13 problems. Each problem is worth 12 points. (For each True/False problem, answer T if it is true, and if it is false, answer F and give an explicit counterexample.)

(1) True or False: Let A and B be subsets of a topological space X. If A and B are connected, then so is $A \cap B$.

Solution: This is false in general. Let $X = \mathbf{R}^2$ be the reals with the standard topology. Let A be the graph of $f(x) = x^2 - 1$ (since $id_{\mathbf{R}}$ and f are continuous, so is $h = id_{\mathbf{R}} \times f : x \mapsto (x, f(x))$, but $h(\mathbf{R}) = A$, so A is connected. Let B be the x-axis (hence B is connected). But $A \cap B = \{(-1, 0), (1, 0)\}$ is not connected.

(2) True or False: Let A and B be subsets of a topological space X. If A and B are compact, then so is $A \cup B$.

Solution: This is true; given any open cover $\{U_i : i \in I\}$ of $A \cup B$, only finitely many, say $\{U_{i_1}, \ldots, U_{i_r}\}$, of the open subsets are needed to cover A and only finitely many, say $\{U_{j_1}, \ldots, U_{j_s}\}$, are needed to cover B, so $\{U_{i_1}, \ldots, U_{i_r}, U_{j_1}, \ldots, U_{j_s}\}$ is a finite subcover that covers $A \cup B$.

(3) True or False: If X is a topological space which is a T_0-space, then $\{x\}$ is closed for every $x \in X$.

Solution: This is false; a topological space X is a T_1-space if and only if every point is closed, but not every T_0-space is a T_1-space. In particular, if $X = \{a, b\}$ and the topology is $\mathcal{T}_X = \{\varnothing, X, \{a\}\}$, then X is a T_0-space (since $b \notin \{a\}$) but not a T_1-space (since $\{a\}$ is not closed).

(4) True or False: If X is a Hausdorff topological space, then $X \times X$ is Hausdorff when given the product topology.

Solution: This is true; let $(a, b), (c, d) \in X$ with $(a, b) \neq (c, d)$. Then either $a \neq c$ or $b \neq d$. Say $a \neq c$. Then since X is Hausdorff, there are disjoint open neighborhoods $a \in U$ and $c \in V$, hence we have disjoint open neighborhoods $(a, b) \in U \times X$ and $(c, d) \in V \times X$. (The proof is similar if $b \neq d$: since X is Hausdorff, there are disjoint open neighborhoods $b \in U$ and $d \in V$, hence we have disjoint open neighborhoods $(a, b) \in X \times U$ and $(c, d) \in X \times V$.)

(5) True or False: If A and B are compact subsets of the reals $\mathbf{R}$ with the standard topology, then $A \cap B$ is also compact.

Solution: This is true; A and B are closed and bounded, hence $A \cap B$ is closed and bounded, hence $A \cap B$ is compact.

(6) True or False: Let C be a closed and bounded subset of a metric space (X, d), where we give X its metric topology. Then C is compact.

Solution: This is false in general. If X is an infinite set with the metric d where $d(x_1, x_2)$ is 1 if $x_1 \neq x_2$ (and $d(x, x) = 0$ for all $x \in X$), then X is bounded and closed but only finite subsets of X are compact. In particular, X is closed and bounded but not compact.

(7) True or False: Let $f : X \to Y$ be a continuous surjective map of topological spaces such that X is Hausdorff. Then Y is also Hausdorff.

Solution: This is false in general. Let $X = Y$ be any set with at least two points, but give X the discrete topology and give Y the indiscrete topology. Let $f(x) = x$ for all $x \in X$ (so f is the identity map). Then f is continuous, X is a T_2-space, but Y is not a T_2-space.

(8) True or False: Let $S = \mathbf{R}$ be the reals. Let R and R' be equivalence relations on S. Then $R \cap R'$ is also an equivalence relation.

Solution: This is true; $(s, s) \in R \cap R'$ for all $s \in S$ since $(s, s) \in R$ for all $s \in S$ (because R is an equivalence relation) and $(s, s) \in R'$ for all $s \in S$ (because R' is an equivalence relation). Thus $R \cap R'$ is reflexive. Likewise, if $(a, b) \in R \cap R'$, then $(b, a) \in R$ and $(b, a) \in R'$ (since both R and R' are symmetric), so $R \cap R'$ is symmetric. And if $(a, b), (b, c) \in R \cap R'$, then $(a, c) \in R$ and $(a, c) \in R'$ (since both R and R' are transitive), so $R \cap R'$ is transitive. Thus $R \cap R'$ is an equivalence relation.

(9) True or False: Let $S = \mathbf{R}$ be the reals. Let R and R' be equivalence relations on S. Then $R \cup R'$ is also an equivalence relation.

Solution: This is false in general. Certainly $R \cup R'$ will be reflexive and symmetric, but it need not be transitive. For example, say $a \sim_R b$ if and only if $a = b$ or $a = 1, b = 2$ or $b = 1, a = 2$. Say $a \sim_{R'} b$ if and only if $a = b$ or $a = 2, b = 3$ or $b = 2, a = 3$. Then $(1, 2) \in R \subset R \cup R'$ and $(2, 3) \in R' \subset R \cup R'$, but $(1, 3) \notin R \cup R'$.

(10) Let $S = \mathbf{R}$ be the reals. Let R be the relation on S where $R = \{(a, b) \in S \times S : ab + 1 = 0\}$.
(a) Is R reflexive? Justify your answer.

Solution: No, R is not reflexive since $a \sim_R a$ means $a^2 + 1 = 0$, so $a \sim_R a$ not only fails, it *never* holds.

(b) Is R symmetric? Justify your answer.

Solution: Yes, R is symmetric since $a \sim_R b$ and $b \sim_R a$ both mean that $ab+1 = 0$ (since multiplication is commutative), so if one holds so does the other.

(c) Is R transitive? Justify your answer.

Solution: No, R is not transitive since $1 \sim_R -1$ and $-1 \sim_R 1$, but $1 \not\sim_R 1$.

(11) Let $S = \mathbf{R}$. Let R be the relation on S where $R = \{(a, b) \in S \times S : \text{either } a \geq 0 \text{ or } b \leq 0\}$.
(a) Is R reflexive? Justify your answer.

Solution: Yes, R is reflexive since $a \sim_R a$ means either $a \geq 0$ or $a \leq 0$, but this holds for all $a \in S$.

(b) Is R symmetric? Justify your answer.

Solution: No, R is not symmetric since $1 \sim_R -1$, but but $-1 \not\sim_R 1$.

(c) Is R transitive? Justify your answer.

Solution: No, R is not transitive, since $-1 \sim_R 0$ and $0 \sim_R 1$, but $-1 \not\sim_R 1$.

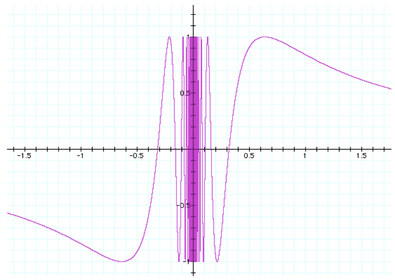
(12) Let (X_1, d_1) and (X_2, d_2) be metric spaces where $|X_1| > 1$ and $|X_2| > 1$. Show that $d = d_1 d_2$, defined by $d((a, b)(c, d)) = d_1(a, c)d_2(b, d)$, is not a metric on $X_1 \times X_2$.

Solution: Let $a, c \in X_1$, $a \neq c$, and $b \in X_2$. Then $d((a, b)(c, b)) = d_1(a, c)d_2(b, b) = 0$, even though $(a, b) \neq (c, b)$, so d is not a metric.

(13) Consider the function $f : \mathbf{R} \to \mathbf{R}$ defined as

$$f(x) = \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Here is a graph:



Assuming that $\mathbf{R}$ has the standard topology, show that $f^{-1}(V)$ is not open, where V is the open interval $(-0.5, 0.5)$. [Hint: show that $0 \in f^{-1}(V)$, but also show that every open neighborhood of 0 contains values of x such that $f(x) \notin V$.]

Solution: Note that $0 \in f^{-1}(V)$ since $f(0) = 0 \in (-0.5, 0.5)$. However $\frac{1}{2n\pi + \frac{\pi}{2}} \notin f^{-1}(V)$ for all integers n, since $f(\frac{1}{2n\pi + \frac{\pi}{2}}) = \sin(2n\pi + \frac{\pi}{2}) = 1 \notin V$. Every open neighborhood of 0 contains $\frac{1}{2n\pi + \frac{\pi}{2}}$ for n large enough, so 0 is in the closure of $(f^{-1}(V))^c$ even though $0 \notin (f^{-1}(V))^c$. Thus $(f^{-1}(V))^c$ is not closed, so $f^{-1}(V)$ is not open.