Exam 1, Thursday, September 27, 2012

Do any 10 of the 14 problems. Each problem is worth 10 points. (For each True/False problem, answer T if it is true, and if it is false, answer F and give an explicit counterexample.)

(1) True or False: Every subset of a topological space is either open or closed.

   Solution: This is false. Let X be a set with at least 2 elements and give it the indiscrete topology (so only X and $\varnothing$ are open). If $x \in X$, then $\{x\}$ is neither open nor closed. Alternatively, take the reals $\mathbf{R}$ with the standard topology. Then $[0,1)$ is neither open nor closed.

(2) True or False: Let $f : X \to Y$ be a map of sets. Then $f(f^{-1}(C)) \subseteq C$ holds for all subsets $C \subseteq Y$.

   Solution: This is true; see Problem 5 on Homework 1.

(3) True or False: If X is a topological space with the discrete topology and if $|X| > 1$, then X is not connected.

   Solution: True: since $|X| > 1$, X has two disjoint nonempty complementary subsets U and V; for example, let $U = \{x\}$ where $x \in X$ and let $V = U^c$. They are both open hence they disconnect X.

(4) True or False: let X be an infinite set. The collection of finite subsets of X, together with X and the empty set, comprises a topology on X.

   Solution: False; an infinite union of finite subsets is not a finite subset, but unions of open sets are supposed to be open.

(5) True or False: For any map $f : X \to Y$ of sets and for all subsets $A, B \subseteq X$, it is true that $f(A \cap B) = f(A) \cap f(B)$.

   Solution: This is false; see Problem 4 on Homework 1.

(6) True or False: If X is a topological space, then there always is a metric on X which gives rise to its topology.

   Solution: This is false; see Problem 1 on Homework 2.

(7) True or False: If X and Y are topological spaces and if $f : X \to Y$ is a constant map (which means that there is a point $y \in Y$ such that $f(x) = y$ for all $x \in X$), then f is continuous.

   Solution: This is true; let $y \in Y$ be the point such that $f(X) = \{y\}$. If $V \subseteq Y$ is open, then $f^{-1}(V) = \varnothing$ if $y \notin V$ and $f^{-1}(V) = X$ if $y \in V$. Thus the preimage of an open set is always open.

(8) True or False: If X is a topological space, then X is both open and closed.

   Solution: This is true; X is open by definition and X is closed since it's the complement of the empty set, which is open by definition.

(9) True or False: If S is a subset of the reals $\mathbf{R}$ with the standard topology such that $\mathrm{Cl}(S)$ is connected, then S is connected.

Solution: This is false; let $S = [0,1) \cup (1,2]$. Then S is not connected but $\mathrm{Cl}(S) = [0,1] \cup [1,2] = [0,2]$ is connected.

(10) Consider the function $f : \mathbf{R} \to \mathbf{R}$ defined as

$$f(x) = \begin{cases} x & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Assuming that $\mathbf{R}$ has the standard topology, find an open subset $V \subseteq \mathbf{R}$ such that $f^{-1}(V)$ is not open.

Solution: Let $V = (0.5, 1.5)$ (in particular, V is a small open neighborhood of the image $f(0)$ of the point $x = 0$ at which f is not continuous). Then $f^{-1}(V) = (0.5, 1.5) \cup \{0\}$, but this is not open since $f^{-1}(V)$ does not contain an open neighborhood of $\{0\}$.

(11) Let X and Y be nonempty topological spaces such that X has the indiscrete topology and Y has the discrete topology. Let $f : X \to Y$ be a map. If f is not constant (i.e., if $|f(X)| > 1$), prove that $f : X \to Y$ is not continuous.

Solution: Suppose that f is not constant. Thus there are two points $a, b \in X$ with $f(a) \neq f(b)$. Then $\{f(a)\}$ is open but $f^{-1}(\{f(a)\})$ is neither empty nor all of X, hence not open. Thus f is not continuous.

(12) Let $A = (0,1] \subset \mathbf{R}$. Find $\mathrm{Cl}(A)$, $\mathrm{Int}(A)$ and $\mathrm{Fr}(A)$ if $\mathbf{R}$ has the standard topology. You do not need to justify your answers.

Solution: We have $\mathrm{Cl}(A) = [0,1]$, $\mathrm{Int}(A) = (0,1)$ and $\mathrm{Fr}(A) = \{0,1\}$.

(13) Let $A = (0,1] \subset \mathbf{R}$. Find $\mathrm{Cl}(A)$, $\mathrm{Int}(A)$ and $\mathrm{Fr}(A)$ if $\mathbf{R}$ has the discrete topology. You do not need to justify your answers.

Solution: We have $\mathrm{Cl}(A) = A$ and $\mathrm{Int}(A) = A$ since A is both open and closed (as is every subset in the discrete topology) and $\mathrm{Fr}(A) = \varnothing$ (since $\mathrm{Fr}(A) = \mathrm{Cl}(A) \cap \mathrm{Cl}(A^c) = A \cap A^c = \varnothing$).

(14) Let $A = (0,1] \subset \mathbf{R}$. Find $\mathrm{Cl}(A)$, $\mathrm{Int}(A)$ and $\mathrm{Fr}(A)$ if $\mathbf{R}$ has the indiscrete topology. You do not need to justify your answers.

Solution: We have $\mathrm{Cl}(A) = \mathbf{R}$ (since $\mathrm{Cl}(A)$ is a closed set that contains A, but the only closed sets are $\varnothing$ and $\mathbf{R}$), $\mathrm{Int}(A) = \varnothing$ (since $\mathrm{Int}(A)$ is an open set contained in A and the only open sets are $\varnothing$ and $\mathbf{R}$), and $\mathrm{Fr}(A) = \mathbf{R}$ (since $\mathrm{Cl}(A) = \mathrm{Int}(A) \cup \mathrm{Fr}(A)$ by Problem 2 of Homework 3, hence here we have $\mathbf{R} = \mathrm{Cl}(A) = \mathrm{Int}(A) \cup \mathrm{Fr}(A) = \varnothing \cup \mathrm{Fr}(A) = \mathrm{Fr}(A)$).