M445/845, Homework 1, due Friday, September 13, 2013

Instructions: Do any three problems.

(1) It is conjectured that there are infinitely many prime pairs $p < q$ with $q - p = 2$. Are there infinitely many prime triples $p < q < r$ with $r - p = 4$? Why or why not? Justify your answer.

(2) Do Problem 2 on p. 17 of the text. Show the steps you used.

(3) The Fibonacci sequence f_i, $i \geq 1$ is defined as $f_1 = f_2 = 1$, with $f_{i+1} = f_i + f_{i-1}$ for $i \geq 2$. Prove that $(f_n, f_{n+1}) = 1$ for all $n \geq 1$.

(4) Consider the Fibonacci sequence f_i, $i \geq 1$, defined above. Show that the Euclidean algorithm takes $n - 2$ steps to evaluate (f_n, f_{n-1}) for $n \geq 3$.

(5) It is known that that the Euclidean algorithm for computing (a, b) takes at most $6\ln(\max(a, b))$ steps. The point of this problem is to show (together with Problem 4) that it can take at least $\ln(\max(a, b))$ steps. Consider the Fibonacci sequence f_i, $i \geq 1$, defined above.
   (a) Prove $f_n \leq 1.62^n$ for all $n \geq 1$.
   (b) Prove that $n - 2 \geq \ln(f_n)$ for $n \geq 3$.

(6) The probability that randomly chosen positive integers a and b are relatively prime is

$$\frac{6}{\pi^2} = 0.6079271\ldots$$

or about 61%(apparently due to Dirichlet in 1849). The point of this problem is to see why this probability is more than 1/2.

Let $d > 0$ be an integer. Let $D_d(N)$ be the probability that the gcd of a randomly chosen ordered pair $(a, b) \in S_N = \{(m, n) : 1 \leq m, n \leq N\}$ is divisible by d; i.e., $D_d(N)$ is the number of ordered pairs in S_N with gcd divisible by d divided by the number of elements of S_N.
   (a) Give an argument for what the value of $\lim_{N\to\infty} D_d(N!)$ is. (You do not need to give a detailed ϵ, δ argument.)
   (b) Let $P(N)$ be the probability that a randomly chosen ordered pair $(a, b) \in S_N$ is divisible by a prime. Show that $P(N) < \sum_p D_p(N)$ when $N \geq 6$, where the sum is over all primes p.
   (c) Let $Q(N)$ be the probability that the gcd of a randomly chosen ordered pair $(a, b) \in S_N$ is 1. Justify why $Q(N) = 1 - P(N)$, and use this with your answers to (a) and (b) to justify why $\lim_{N\to\infty} Q(N!) > \frac{1}{2}$.