M417 Final Exam **Solutions** **May 7, 2004**

(1) Prove that a group G is a union of proper subgroups if and only if G is not cyclic.

Answer: If G is cyclic we must show that G is not the union of proper subgroups. But say $G = <g>$; if G were then union of proper subgroups, then g must be in a proper subgroup, but any subgroup of G containing g contains G, hence is not proper. Conversely, say G is not cyclic. Then $<g>$ is a proper subgroup for every element $g \in G$, and clearly G is the union, taken over all $g \in G$, of the subgroups $<g>$.

(2) (a) Define what it means for a subgroup N of a group H to be normal.

(b) Let $f : G \to H$ be a homomorphism of groups. Let N be a normal subgroup of H. Prove that $f^{-1}(N)$ is a subgroup of G and that it is normal in G.

Answer: (a) We say a subgroup $N < H$ is normal if $hNh^{-1} = N$ for every $h \in H$. (Alternatively, if $hnh^{-1} \in N$ for every $n \in N$ and every $h \in H$.)

(b) First, $e_G \in f^{-1}(N)$ since $f(e_G) = e_H \in N$, so $f^{-1}(N) \neq \emptyset$. Next, if $x, y \in f^{-1}(N)$, then $f(xy) = f(x)f(y) = e_H e_H = e_H$, so $xy \in f^{-1}(N)$, and finally if $x \in f^{-1}(N)$, then $f(x^{-1}) = f(x)^{-1} = e_H^{-1} = e_H$, so $x^{-1} \in f^{-1}(N)$. Thus $f^{-1}(N)$ is a subgroup of G. To see that $f^{-1}(N)$ is normal, let $x \in f^{-1}(N)$ and let $g \in G$. Then $f(gxg^{-1}) = f(g)f(x)f(g)^{-1}$ is in N since $f(x) \in N$ and N is normal, so $gxg^{-1} \in f^{-1}(N)$, hence $f^{-1}(N)$ is normal.

(3) (a) Define the center $Z(G)$ of a group G.

(b) Suppose $x \in G$ is an element of order 2 in a group G. If x is the only element of order 2 in G, prove that x is in the center $Z(G)$ of G.

Answer: (a) $Z(G) = \{g \in G | gx = xg \text{ for all } x \in G\}$

(b) Say x is the only element of G of order 2. Note that $(gxg^{-1})^2 = gxg^{-1}gxg^{-1} = gxxg^{-1} = ge_Gg^{-1} = gg^{-1} = e_G$, so either $gxg^{-1} = e_G$ or gxg^{-1} has order 2. The first case can't happen, since $gxg^{-1} = e_G$ implies that $x = g^{-1}g = e_G$, which doesn't have order 2. And the second case implies that $gxg^{-1} = x$, hence $xg = gx$ for all $g \in G$, so $x \in Z(G)$.

(4) (a) Define the centralizer $C_G(x)$ of an element x in a group G.

(b) Determine $|C_G(x)|$ if $G = S_5$ and $|x| = 6$. [Hint: determine the order of the orbit of x under conjugation and use the orbit-stabilizer theorem and use the fact, which you may assume, that $\text{stab}_G(x) = C_G(x)$.]

Answer: (a) $C_G(x) = \{g \in G | gx = xg\}$

(b) If we write x as a product of disjoint cycles, then x must be a product of a 2-cycle and a 3-cycle, since no other product of disjoint cycles in G has order 6. Now all elements of G which are a product of a disjoint 2-cycle and 3-cycle are conjugate, and so form a single orbit under the action of G on G by conjugation. There are $2\binom{5}{3}$ different 3-cycles in G, hence also $2\binom{5}{3} = 20$ elements which are a product of a disjoint 2-cycle and 3-cycle. Thus $|\text{orb}_G(x)| = 20$. But $\text{stab}_G(x) = C_G(x)$, and $\text{stab}_G(x)\text{orb}_G(x) = |G| = 5! = 120$, so $|C_G(x)| = 120/20 = 6$. (Aside: Since we know $<x> \in C_G(x)$ and since $<g>$ has order 6, this shows that $<g> = C_G(x)$.)

(5) Let $a_1 = 1$ and for $k \geq 1$, let $a_{k+1} = a_k + 2^k$. Prove that $a_k = 2^k - 1$ for all $k \geq 1$.

Answer: First, $a_1 = 1 = 2^1 - 1$, so the claim is true for $k = 1$. Now assume that the claim is true for some k. Then $a_{k+1} = a_k + 2^k = (2^k - 1) + 2^k = 2 * 2^k - 1 = 2^{k+1} - 1$, which shows that the claim holds for $k + 1$. This proves the claim holds for all $k \geq 1$ by induction.

(6) For which integers $n > 1$ is there exactly one homomorphism $f : \mathbf{Z}_n \to \mathbf{Z}_n$ which is not an isomorphism? You may assume for each $m \in \mathbf{Z}_n$, that there is a homomorphism $f : \mathbf{Z}_n \to \mathbf{Z}_n$ with $f(1) = m$ (and hence that there are exactly n homomorphisms $f : \mathbf{Z}_n \to \mathbf{Z}_n$). Justify your answer.

Answer: There is exactly one homomorphism $f : \mathbf{Z}_n \to \mathbf{Z}_n$ which is not an isomorphism if and only if n is prime. Certainly, for every $n > 1$, we have the homomorphism that sends every element to 0. This is not surjective, hence not an isomorphism, so there is always at least one nonisomorphic homomorphism. Now note that a homomorphism in our situation (since we are mapping a finite group to itself) is injective if and only if it is surjective, and f is surjective if and only if $f(1)$ is a generator of $\mathbf{Z}_n$. For there to be exactly one nonisomorphic homomorphism, it must therefore be true that every nonzero element of $\mathbf{Z}_n$ generates (if m is nonzero but does not generate $\mathbf{Z}_n$, then the homomorphism for which $f(1) = m$ is not an isomorphism, and thus there are at least two nonisomorphic homomorphisms). Thus every integer from 1 to $n - 1$ is relatively prime to n, hence n is prime. And if n is prime, then $f(1)$ generates as long as $f(1) \neq 0$, so there is only one nonisomorphic homomorphism, this being the one with $f(1) = 0$.

(7) Define $f : \mathbf{Z}_{77} \to \mathbf{Z}_7 \times \mathbf{Z}_{11}$ by $f(x) = (x \bmod 7, x \bmod 11)$. Define $g : \mathbf{Z}_7 \times \mathbf{Z}_{11} \to \mathbf{Z}_{77}$ by $g((x, y)) = 11x + 7y \bmod 77)$. Define $h : \mathbf{Z}_7 \times \mathbf{Z}_{11} \to \mathbf{Z}_7 \times \mathbf{Z}_{11}$ by $h = f \circ g$. Find an element $(m, n) \in \mathbf{Z}_7 \times \mathbf{Z}_{11}$ such that $h((m, n)) = (1, 1)$.

Answer: Note that $h((m, n)) = (11m \bmod 7, 7n \bmod 11)$, so we are looking for a solution to the equations $11m \bmod 7 = 1$ and $7n \bmod 11 = 1$. The solution is $x = 2$ and $y = 8$.

Extra Credit: Let k be a positive integer and let G be a group of order $2(2k + 1)$. Prove that 2 divides $|Z(G)|$ if and only if G has a unique element of order 2. (You may assume Problem 3 in doing this problem.)

Answer: If G has a unique element of order 2, then by Problem 3, we know the element is in the center, so by Lagrange's Theorem, the order of the center is even. Conversely, say $|Z(G)|$ is even. Then by Cauchy's Theorem, $Z(G)$ has an element x of order 2. Since $|G| = 2(2k + 1)$, $<x>$ is a Sylow 2-subgroup. Since it is in the center, $<x>$ is always its own conjugate, so it is normal. Since by Sylow's theorems the Sylow 2-subgroups are all conjugate, this means that $<x>$ is the only Sylow 2-subgroup, hence x is the only element of G of order 2 (any other element would generate another subgroup of order 2, which would then be another Sylow 2-subgroup). Thus x is the only element of G of order 2.