M417 Homework 4 Solutions Spring 2004

(1) Determine the orders of the groups of symmetries of the Platonic solids: The tetrahedron, as we found in class, has symmetry group of order 12. Likewise, the cube's has order 24. The octahedron has 6 vertices, with 4 faces at each vertex, so the order is $6 \times 4 = 24$. The dodecahedron has 20 vertices, with 3 faces at each vertex, so the order is $20 \times 3 = 60$. The icosahedron has 12 vertices, with 5 faces at each vertex, so the order is $12 \times 5 = 60$.

(2) If $x^2 = e$ for every element x of a group G, show that G is abelian: given $a, b \in G$, we must show $ab = ba$. But $aabb = a^2b^2 = ee = e = (ab)^2 = abab$, so cancelling gives $ab = ba$.

(3) # 16, p. 38: Label the H's, in order, by the integers, using subscripts. Let t_d be the translation $t_d(H_i) = H_{i+d}$. Let h be the reflection across the horizontal line through the center of the H's. Let r_d be the rotation by 180°, centered on H_d, let r'_d be the rotation by 180°, centered between H_d and H_{d+1}, let v_d be the reflection across the vertical line through the center of H_d, and let v'_d be the reflection across the vertical line midway between H_d and H_{d+1}. Any symmetry takes H_0 somewhere, say to H_d, first either rotating H_0 by a half turn, or flipping H_0 across either its horizontal or vertical axis of symmetry. And once you know how H_0 was moved, you know how all the other H's were moved too. Thus every symmetry is either $t_d r_0$, $t_d v_0$, or $t_d h$. (These are just the symmetries we found above, since $t_{2d} r_0 = r_d$, $t_{2d+1} r_0 = r'_d$, $t_{2d} v_0 = v_d$, $t_{2d+1} v_0 = v'_d$, and $v_0 = h r_0$. Thus every symmetry can be obtained using just r_0, h, and t_d, $d \in \mathbf{Z}$.) But $r_0 t_{2d} = r_{-d}$ while $t_{2d} r_0 = r_d$, so the symmetries don't always commute.

(4) # 50, p. 71: if a subgroup contains positive integers, a and b then it contains every integer linear combination of a and b, hence it contains $k = gcd(a, b)$. Since $< k >$ contains a and b, $< k >$ is the smallest subgroup of $\mathbf{Z}$ containing a and b. Thus the answer for: (a) is $k = gcd(8, 14) = 2$; (b) is $k = gcd(8, 13) = 1$; and (c) is $k = gcd(6, 15) = 3$. For (d), the same reasoning gives $k = gcd(|m|, |n|)$, unless $m = n = 0$, in which case $gcd(|m|, |n|)$ is undefined but we can take $k = 0$. For (e), any subgroup that contains 12 and 18 contains 6, and any subgroup that contains 6 and 45 contains 3, while $< 3 >$ contains 12, 18 and 45, so the answer is $k = 3$. Note that again $k = gcd(2, 18, 45)$.

(5) # 52, p. 71: Consider $e \neq x \in G$. Since G is finite, we know there exist integers $m < n$ such that $x^m = x^n$. Cancelling gives $e = x^k$, for $k = n - m > 1$ (since $k = 1$ implies $x = e$). This shows that $x^k = e$ has solutions $k > 1$. Replace k by the least such solution. Thus we may assume that $k > 1$ and that $x^k = e$, but that $x^i \neq e$ for $1 \leq i < k$. Let p be any prime dividing k, and define m by $pm = k$. Let $y = x^m$. Then $y^p = x^{pm} = x^k = e$, but for $0 < j < p$, $y^j = x^{jm}$ is not e since $0 < jm < k$. Thus $|y| = p$.