M417 Exam 2 Solutions **April 21, 2004**

Instructions: Do any three problems. Each problem counts 16.666... points. You may select a fourth problem for up to 5 points of extra credit, but if you do four or more problems, you must specify which are the three primary problems, which is for extra credit, and which are not to be graded.

(1) Find the centralizer of $x = (1\ 2\ 3\ 4)$ in S_4. Justify your answer.

Answer: Having $g \in C_{S_4}(x)$ means $gxg^{-1} = x$. But $gxg^{-1} = (g(1)\ g(2)\ g(3)\ g(4))$, hence $(g(1)\ g(2)\ g(3)\ g(4)) = (1\ 2\ 3\ 4)$, so g must be a cyclic permutation of 1, 2, 3 and 4. Thus $g \in < x >$, so $< x > \subset C_{S_4}(x)$. Since x commutes with its own powers, we also have $C_{S_4}(x) \subset < x >$, so $C_{S_4}(x) = < x >$.

(2) Show that a subgroup of index 2 is always a normal subgroup. Do not assume that the groups are finite.

Answer: Let A have index 2 in a group G. Let g be any element of G. If $g \in A$, then $gA = A = Ag$. If g is not in A, then, since there are exactly two left cosets of A in G and g is not in A, they must be A and gA. Since left cosets are disjoint, we know $gA = G - A$. But right cosets are also disjoint, so $Ag = G - A$. Hence$Ag = G - A = gA$. Thus $gA = Ag$ for all $g \in G$, so A is normal.

(3) Let $f : \mathbf{Z}_{143} \to \mathbf{Z}_{11} \times \mathbf{Z}_{13}$ be the homomorphism defined by $f(x) = (x \bmod 11, x \bmod 13)$. Find an element $x \in \mathbf{Z}_{143}$ such that $f(x) = (7, 4)$. Show how you obtain your answer.

Answer: So we must simultaneously solve $x \bmod 11 = 7$ and $x \bmod 13 = 4$. The first equation tells us that $x = 7 + 11t$ for some t, so the second equation then gives $7 + 11t = 4 + 13s$ for some s, hence $3 = 13s - 11t$. Either using the Euclidean Algorithm or by inspection, we find $1 = 13(-5) - 11(-6)$, hence $3 = 13(-15) - 11(-18)$, so we can take $x = 7 + 11(-18) = -191$. Modulo 143, this is $286 - 191 = 95$, so $x = 95$.

(4) Let N and M be normal subgroups of a group G. Prove that the subgroup $N \cap M < G$ is normal.

Answer: It is enough to show $g(N \cap M)g^{-1} \subset N \cap M$ for all $g \in G$. But if $y \in g(N \cap M)g^{-1}$, then $y = gxg^{-1}$ for some $x \in N \cap M$. Now, y is in N since N is normal and $x \in N$, and y is in M since M is normal and $x \in M$, so y is in the intersection $N \cap M$, as we needed to show.

(5) Determine the number of different homomorphisms $f : \mathbf{Z}_2 \times \mathbf{Z}_3 \to \mathbf{Z}_3 \times \mathbf{Z}_5$.

Answer: First of all, $\mathbf{Z}_2 \times \mathbf{Z}_3$ is isomorphic to $\mathbf{Z}_6$, and $\mathbf{Z}_3 \times \mathbf{Z}_{15}$ is isomorphic to $\mathbf{Z}_{15}$, so we just need to count the number of homomorphisms $f : \mathbf{Z}_6 \to \mathbf{Z}_{15}$. But each homomorphism is completely determined by the value of $f(1)$, and, as on the homework, all we need is that 15 divides $6f(1)$. But 15 divides $6f(1)$ if and only if 5 divides $f(1)$, so $f(1)$ must be either 0, 5 or 10. So there are 3 homomorphisms.

(6) Determine the number of different ways to color an equilateral triangle using 7 colors, where we would regard two congruent equilateral triangles as being colored the same if one could be picked up and placed on the other to make the colors match up.

Answer: You can do this using Burnside's Theorem, as we did in class. Here's a seat-of-the-pants method. There are 7 ways to color the triangle if all the sides have the same color. There is only 1 way to color the triangle using three given colors if each of the three colors is used, and there are 35 ways to choose a subset of 3 colors from a set of 7. Finally, there are two ways to color a triangle using two given colors, if both are used (one way uses one color on just one side, the other way uses that color on two sides). There are 21 ways to choose the two colors, and hence 42 two-color colorings. This gives $7 + 35 + 42 = 84$ colorings. Now we do it using Burnside. Calling the sides of the triangle a, b and c, there are 7 ways to color each side, for 7^3 colorings (where we keep track of which side got which color). But what we really want is the number of orbits under the action of D_3. The number of colorings fixed by either of the two rotations is 7. The number of colorings fixed by any of the three reflections is 7^2 (any one color for the sides that get reflected onto each other, and any one for the third side, which gets reflected onto itself). The identity fixes all 7^3 sides. By Burnside's Theorem, the number of orbits is the sum $7 + 7^2 + 7^3$ divided by $|D_3|$, giving $(2 * 7 + 3 * 7^2 + 7^3)/6 = 84$.