[1] Consider the systems of equations:

$$x + (k+5)y = h$$
$$x + y = 3$$

(a) Determine all values of k and h such that the system has no solutions. Explain your answer.

Subtract equation 2 from equation 1 to get $(k+4)y = h-3$. This has no solution if and only if $k = -4$ and $h \neq 3$, since in that case $(k+4)y = h-3$ becomes $0 = 1$, while if $k \neq -4$ we have $y = (h-3)/(k+4)$ and $x = 3-(h-3)/(k+4)$, and if $k = -4$ and $h = 3$, then $x = 3 - y$ gives a solution for any value of y.

(b) Determine all values of k and h such that the system has infinitely many solutions. Explain your answer.

We just saw that we have a free variable if and only if $k = -4$ and $h = 3$.

(c) Determine all values of k and h such that the system has exactly one solution. Explain your answer.

All that remains is the case that $k \neq -4$ for which we have $y = (h-3)/(k+4)$ and $x = 3-(h-3)/(k+4)$.

[2] Let A be an $n \times n$ matrix.

(a) If A^2 is the 0 matrix, prove that $\lambda = 0$ is an eigenvalue of A and that it is the only eigenvalue of A.

Say that $Av = \lambda v$, where v is an eigenvector, hence $v \neq 0$. Then $0 = A^2 v = \lambda A v = \lambda^2 v$, so $v \neq 0$ implies $\lambda^2 = 0$, hence $\lambda = 0$.

(b) If A^2 is the 0 matrix and A is diagonalizable, prove that A is the 0 matrix.

There is an invertible matrix P such that $P^{-1}AP = D$ is diagonal. Thus $0 = P^{-1}A^2P = (P^{-1}AP)(P^{-1}AP) = D^2$, so $D = 0$, so $A = PDP^{-1} = 0$.

[3] In this problem, R_A is the reduced row echelon form of the matrix A. For each of the following statements, circle T if it is true and F if it is false. In addition, if it is false, write down a specific matrix A for which the statement does not hold.

(a) True: For every matrix A, $\mathrm{Nul}(A) = \mathrm{Nul}(R_A)$.

(b) True: For every matrix A, $\mathrm{Row}(A) = \mathrm{Row}(R_A)$.

(c) False: For every matrix A, $\mathrm{Col}(A) = \mathrm{Col}(R_A)$. Let $A = (1,1)^T$. Then $\mathrm{Col}(A) = \mathrm{Span}((1,1)^T)$, but $\mathrm{Col}(R_A) = \mathrm{Span}((1,0)^T)$, and these are different.

(d) False: Every $n \times n$ matrix A has the same characteristic polynomial as R_A. Let $A = 2I$. Then the polynomial for A is $(2-t)^n$, but $R_A = I$ has $(1-t)^n$.

(e) False: For every $n \times n$ matrix A, $\det(A) = \det(R_A)$. Again, use $A = 2I$ and $A = I$: $\det(A) = 2^n$, but $\det(R_A) = 1$.

(f) True: An $n \times n$ matrix A is invertible if and only if R_A is.

(g) False: An $n \times n$ matrix A is diagonalizable if and only if R_A is. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. Then $R_A = I$ is diagonalizable (even diagonal), but A is not diagonalizable, since the eigenvalue $\lambda = 1$ has algebraic multiplicity 2 but geometric multiplicity only 1.

(h) True: For every matrix A, $\mathrm{rank}(A) = \mathrm{rank}(R_A)$.

(i) True: For every matrix A, $\dim \mathrm{Row}(A) = \dim \mathrm{Row}(A^T)$.

(j) True: For every matrix A, $\mathrm{Nul}(A) = (\mathrm{Row}(A))^\perp$.

[4] Let H be the subset $\{[a+1, b]^T : a, b \in \mathbf{R}\}$ of $\mathbf{R}^2$. Let W be the subset $\{[1, b]^T : b \in \mathbf{R}\}$ of $\mathbf{R}^2$.

(a) Is W a subspace of $\mathbf{R}^2$? Explain why or why not.

No, W is not a subspace, since it is not closed under vector addition or scalar multiplication. For example, $[1,0]^T \in W$, but $2[1,0]^T \notin W$.

(b) Is H a subspace of $\mathbf{R}^2$? Explain why or why not.

Yes, H is a subspace, since in fact $H = \mathbf{R}^2$.

[5] Let H be the span of the vectors $[0,1,0,1]^T$, $[1,0,1,0]^T$, $[0,1,1,0]^T$, and $[1,0,0,1]^T$. For each part, show your work or explain your answer.

(a) Find the dimension of H.

The dimension of H is just the rank of the matrix A whose columns are the given vectors. By row reduction, we find that $\text{rank}(A) = 3$. Note that it is not enough here to find $\det(A)$. If $\det(A)$ were not 0, then we would know that A has rank 4, but here $\det(A) = 0$, so all we know is that the rank of A is less than 4.

(b) Determine if $[1,1,1,1]^T$ is in H.

Here we try to solve $Ax = v$ where $v = [1,1,1,1]^T$. By row reducing we find that the system is consistent, so $v \in H$. In fact, $x = [1,1,0,0]^T$ is a solution.

(c) Find an orthogonal basis for H.

Note that $[0,1,0,1]^T$ and $[1,0,1,0]^T$ are already orthogonal. By Gram-Schmidt, we get from the third vector $[0,1,1,0]^T$, the vector $(1/2)[-1,1,1,-1]^T$. Since the first three vectors give a basis for H, the vectors $[0,1,0,1]^T$, $[1,0,1,0]^T$ and $(1/2)[-1,1,1,-1]^T$ give an orthogonal basis, as does $[0,1,0,1]^T$, $[1,0,1,0]^T$ and $[-1,1,1,-1]^T$.

(d) Find $\text{proj}_H \mathbf{v}$ for $\mathbf{v} = [1,2,3,4]^T$.

Since we have an orthogonal basis for H, we can use the formula, which gives

$$\text{proj}_H \mathbf{v} = (6/2)[0,1,0,1]^T + (4/2)[1,0,1,0]^T + 0[-1,1,1,-1]^T = [2,3,2,3]^T.$$