[1] Consider the systems of equations:

\[
\begin{align*}
  x + (k + 5)y &= h \\
  x + y &= 3
\end{align*}
\]

(a) Determine all values of \(k\) and \(h\) such that the system has no solutions. Explain your answer.

Subtract equation 2 from equation 1 to get \((k+4)y = h - 3\). This has no solution if and only if \(k = -4\) and \(h \neq 3\), since in that case \((k+4)y = h - 3\) becomes \(0 = 1\), while if \(k \neq -4\) we have \(y = (h-3)/(k+4)\) and \(x = 3 - (h-3)/(k+4)\), and if \(k = -4\) and \(h = 3\), then \(x = 3 - y\) gives a solution for any value of \(y\).

(b) Determine all values of \(k\) and \(h\) such that the system has infinitely many solutions. Explain your answer.

We just saw that we have a free variable if and only if \(k = -4\) and \(h = 3\).

(c) Determine all values of \(k\) and \(h\) such that the system has exactly one solution. Explain your answer.

All that remains is the case that \(k \neq -4\) for which we have \(y = (h-3)/(k+4)\) and \(x = 3 - (h-3)/(k+4)\).

[2] Let \(A\) be an \(n \times n\) matrix.

(a) If \(A^2\) is the 0 matrix, prove that \(\lambda = 0\) is an eigenvalue of \(A\) and that it is the only eigenvalue of \(A\).

Say that \(Av = \lambda v\), where \(v\) is an eigenvector, hence \(v \neq 0\). Then \(0 = A^2v = \lambda Av = \lambda^2v\), so \(v \neq 0\) implies \(\lambda^2 = 0\), hence \(\lambda = 0\).

(b) If \(A^2\) is the 0 matrix and \(A\) is diagonalizable, prove that \(A\) is the 0 matrix.

There is an invertible matrix \(P\) such that \(P^{-1}AP = D\) is diagonal. Thus \(0 = P^{-1}A^2P = (P^{-1}AP)(P^{-1}AP) = D^2\), so \(D = 0\), so \(A = PDP^{-1} = 0\).

[3] In this problem, \(R_A\) is the reduced row echelon form of the matrix \(A\). For each of the following statements, circle \(T\) if it is true and \(F\) if it is false. In addition, if it is false, write down a specific matrix \(A\) for which the statement does not hold.

(a) True: For every matrix \(A\), \(\text{Nul}(A) = \text{Nul}(R_A)\).

(b) True: For every matrix \(A\), \(\text{row}(A) = \text{row}(R_A)\).

(c) False: For every matrix \(A\), \(\text{Col}(A) = \text{Col}(R_A)\). Let \(A = (1,1)^T\). Then \(\text{Col}(A) = \text{Span}((1,1)^T)\), but \(\text{Col}(R_A) = \text{Span}((1,0)^T)\), and these are different.

(d) False: Every \(n \times n\) matrix \(A\) has the same characteristic polynomial as \(R_A\). Let \(A = 2I\). Then the polynomial for \(A\) is \((2-t)^n\), but \(R_A = I\) has \((1-t)^n\).

(e) False: For every \(n \times n\) matrix \(A\), \(\det(A) = \det(R_A)\). Again, use \(A = 2I\) and \(A = I\): \(\det(A) = 2^n\), but \(\det(R_A) = 1\).

(f) True: An \(n \times n\) matrix \(A\) is invertible if and only if \(R_A\) is.

(g) False: An \(n \times n\) matrix \(A\) is diagonalizable if and only if \(R_A\) is. Let \(A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\). Then \(R_A = I\) is diagonalizable (even diagonal), but \(A\) is not diagonalizable, since the eigenvalue \(\lambda = 1\) has algebraic multiplicity 2 but geometric multiplicity only 1.

(h) True: For every matrix \(A\), \(\text{rank}(A) = \text{rank}(R_A)\).

(i) True: For every matrix \(A\), \(\text{dim}(\text{row}(A)) = \text{dim}(\text{row}(A^T))\).

(j) True: For every matrix \(A\), \(\text{Nul}(A) = (\text{row}(A))^\perp\).

[4] Let \(H\) be the subset \(\{(a + 1,b)^T : a, b \in \mathbb{R}\}\) of \(\mathbb{R}^2\). Let \(W\) be the subset \(\{(1,b)^T : b \in \mathbb{R}\}\) of \(\mathbb{R}^2\).

(a) Is \(W\) a subspace of \(\mathbb{R}^2\)? Explain why or why not.

No, \(W\) is not a subspace, since it is not closed under vector addition or scalar multiplication. For example, \([1,0]^T \in W\), but \([2,0]^T \notin W\).

(b) Is \(H\) a subspace of \(\mathbb{R}^2\)? Explain why or why not.
Yes, $H$ is a subspace, since in fact $H = \mathbb{R}^2$.

[5] Let $H$ be the span of the vectors $[0, 1, 0, 1]^T$, $[1, 0, 1, 0]^T$, $[0, 1, 1, 0]^T$, and $[1, 0, 0, 1]^T$. For each part, show your work or explain your answer.

(a) Find the dimension of $H$.

The dimension of $H$ is just the rank of the matrix $A$ whose columns are the given vectors. By row reduction, we find that $\text{rank}(A) = 3$. Note that it is not enough here to find $\det(A)$. If $\det(A)$ were not 0, then we would know that $A$ has rank 4, but here $\det(A) = 0$, so all we know is that the rank of $A$ is less than 4.

(b) Determine if $[1, 1, 1, 1]^T$ is in $H$.

Here we try to solve $Ax = v$ where $v = [1, 1, 1, 1]^T$. By row reducing we find that the system is consistent, so $v \in H$. In fact, $x = [1, 1, 0, 0]^T$ is a solution.

(c) Find an orthogonal basis for $H$.

Note that $[0, 1, 0, 1]^T$ and $[1, 0, 1, 0]^T$ are already orthogonal. By Gram-Schmidt, we get from the third vector $[0, 1, 1, 0]^T$, the vector $(1/2)[-1, 1, 1, -1]^T$. Since the first three vectors give a basis for $H$, the vectors $[0, 1, 0, 1]^T$, $[1, 0, 1, 0]^T$ and $(1/2)[-1, 1, 1, -1]^T$ give an orthogonal basis, as does $[0, 1, 0, 1]^T$, $[1, 0, 1, 0]^T$ and $[-1, 1, 1, -1]^T$.

(d) Find $\text{proj}_H v$ for $v = [1, 2, 3, 4]^T$.

Since we have an orthogonal basis for $H$, we can use the formula, which gives

$$\text{proj}_H v = (6/2)[0, 1, 0, 1]^T + (4/2)[1, 0, 1, 0]^T + 0[-1, 1, 1, -1]^T = [2, 3, 2, 3]^T.$$