[1] Let S be the set $\{a, b, c\}$.

(a) Define a relation on the set S by putting checkmarks in a labeled tic-tac-toe grid, as on the board. Check as many squares as possible, such that the relation you define is NOT reflexive.

Solution: You just don't want to check all of the diagonal positions. As long as you skip at least one of the diagonal positions (i.e., either (a, a), or (b, b) or (c, c)), the relation won't be reflexive, even if you check the other two daigonal positions. So just check eight of the nine positions, skipping one on the diagonal.

(b) This time let the set S be the set of all people. Say person A is related to person B if A and B share a grandparent, but not a parent (i.e., if A and B are cousins but not siblings). For each of the properties reflexivity, symmetry and transitivity, determine whether or not the property holds for this relation. Justify your answer in each case.

Solution: The relation is not reflexive, since a person has a parent in common with himself. The relation is symmetric: if A and B have a grandparent in common bit not a parent, then the sameis true for B and A. It is not transitive, since A and C could be siblings, and B could be a cousin. Thus A is related to B, and B to C, but A and C are not related since they have a parent in common. Also, A and B could be cousins (on the mother's side for B), and B and C could be cousins (on the father's side of B), so A and C need not have a grandparent in common.

[2] Prove the formula $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for each $n \geq 1$. (I.e., prove that the sum of the first n odd integers equals n^2.) Include each step of the proof in your answer.

Solution: First check the base case: $1 = 1^2$. Now assume for some $k \geq 1$ that $1 + 3 + 5 + \cdots + (2k - 1) = k^2$. Then $1 + 3 + 5 + \cdots + (2(k+1) - 1) = 1 + 3 + 5 + \cdots + (2k - 1) + (2(k+1) - 1) = k^2 + (2(k + 1) - 1) = k^2 + 2k + 1 = (k + 1)^2$. Thus $1 + 3 + 5 + \cdots + (2k - 1) = k^2$ implies $1+3+5+\cdots+(2(k+1)-1) = (k+1)^2$. It now follows by induction that $1+3+5+\cdots+(2n-1) = n^2$ for each $n \geq 1$.

[3] Let $a_1 = 3$, $a_2 = 5$, and define $a_{n+1} = 3a_n - a_{n-1}$ for $n \geq 2$. Prove that $a_k > 2^k$ for each integer $k \geq 1$.

Semi-solution: This one turned out to be more involved than I had intended. Unfortunately, I did not notice this until after the exam. What I intended was for this problem to follow the pattern of those we had done in class, so I gave full credit for those who tried to do it that way, thereby showing that they understood how to handle those examples we had studied. However, it's instructive to recognize that this problem doesn't quite work the same way. If we change the problem so that a_{n+1} is defined with a plus sign (i.e., so that $a_{n+1} = 3a_n + a_{n-1}$), then it does follow the pattern. Here's how the proof goes with the plus sign.

First we check that $a_1 > 2^1$ and that $a_2 > 2^2$, both of which are true. Now we show for $k \geq 2$ that $a_k > 2^k$ and $a_{k-1} > 2^{k-1}$ together imply $a_{k+1} > 2^{k+1}$: $a_{k+1} = 3a_k + a_{k-1} > 3(2^k) + 2^{k-1} \geq 2(2^k) = 2^{k+1}$. It now follows that $a_n > 2^n$ for each integer $n \geq 1$.

The problem when we have the minus sign is that knowing $a_k > 2^k$ and $a_{k-1} > 2^{k-1}$ is not enough to guarantee that $3a_k - a_{k-1} > 3(2^k) - 2^{k-1}$. [As an example to see why $x > b$ and $y > c$ do not imply $3x - y > 3b - c$, just take $x = 10$, $y = 9$, $b = 9$ and $c = 5$.] On the homework, I ask you to give a proof of this problem with the minus sign.

[4]
(a) Use Euclid's method to find gcd(357, 918).

Solution: $918 = 357 * 2 + 204$, $357 = 204 * 1 + 153$, $204 = 153 * 1 + 51$, $153 = 51 * 3 + 0$. Thus $\gcd(357, 918) = 51$.

(b) Show how to use your work in (a) to find an integer solution to $918x + 357y = \gcd(357, 918)$.

Solution: Let $n = 918$ and $m = 357$. Thus $918 = 357 * 2 + 204$ gives $n - 2m = 204$, so from $357 = 204 * 1 + 153$ we get $153 = m - 204 * 1 = m - (n - 2m) = 3m - n$. Finally, from $204 = 153 * 1 + 51$ we get $51 = 204 - 151 = (n - 2m) - (3m - n) = 2n - 5m$. Thus $x = 2$ and $y = -5$ is a solution to $918x + 357y = \gcd(357, 918)$.

(c) Find the least positive integer y such that $918x + 357y = \gcd(357, 918)$ has a solution where x also is an integer. Justify your answer.

Solution: From Proposition 5 on page 34, the set of all solutions x and y to $918x + 357y = \gcd(357, 918)$ is $x = 2 + km/g$ and $y = -5 - kn/g$, where $g = \gcd(357, 918)$, and k ranges through all integers. Since $n/g = 918/51 = 18$, so the set of all y's that occur in a solution is $y = -5 - k18$. The least positive y is the one we get if we take $k = -1$; i.e., the least positive y is 13.

[5]
(a) Explain why the least positive integer linear combination $111x + 74y$ of 111 and 74 is 37.

Solution: The gcd of 111 and 74 is 37, and we know by Bezout that 111x+74y=37 has a solution. Thus 37 is a positive integer linear combination of 111 and 74. We also know that 37 divides every integer linear combination of 111 and 74. Hence there can't be any positive integer linear combination of 111 and 74 smaller than 37, because 37 wouldn't divide it.

(b) Let k be an integer. Justify why $111x + 74y = k$ has a solution if and only if $37|k$.

Solution: First assume that $111x + 74y = k$ has a solution. Since 37 divides 111 and 74, it also divides every integer linear combination $111x + 74y = k$ of 111 and 74. Thus $37|k$. Conversely, assume $37|k$; i.e., assume that $k = 37d$ for some integer d. We know that there is a solution a and b in integers to $111a + 74b = 37$, since 37 is the gcd of 111 and 74. Thus $111da + 74db = d(111a + 74b) = 37d = k$; i.e., $x = ad$ and $y = bd$ is a solution to $111x + 74y = k$.