

## Project

### Population Models

In Section 6.4 of the text, the following population model is analyzed:

$$y'(t) = ky(t), \tag{1}$$

where  $y(t)$  is the population size at time  $t$  and  $k$  is a constant. This is called a Malthusian model. Recalling the definition of  $y'(t)$ , if  $y(t)$  satisfies (1), then

$$y(t + 1) - y(t) \approx \lim_{\Delta t \rightarrow 0} \left[ \frac{y(t + \Delta t) - y(t)}{\Delta t} \right] = y'(t) = ky(t), \tag{2}$$

so  $k$  tells us approximately by what percentage the population increases in each unit of time. For example, if  $k = 1$ , then after each unit of time, the population increases by approximately 100%; i.e. it approximately doubles after each unit of time. Thus  $k$  is closely related to the effective birthrate (effective birthrate = birth rate–death rate) for the population. In this project, you will analyze population models that incorporate details not captured by the model in (1).

Implicit in (1) is the assumption that the birthrate is constant. In nature, this is not typical. Many species exhibit seasonal breeding cycles. Their effective birthrate may also depend on longer term weather or geological patterns. To take this into account, we can allow  $k$  to be dependent on time:

$$y'(t) = k(t)y(t),$$

where  $k(t)$  is a given function. This is a Malthusian model with a time dependent birthrate.

Also implicit in (1) is the assumption that there are unlimited resources. A popular model that takes into account limited resources is the logistic model:

$$y'(t) = k(t)y(t) \left( 1 - \frac{y(t)}{R} \right).$$

Assuming that  $0 \leq y(t) \leq R$ , the function  $k(t)$  is related to the effective birthrate of the population if there are unlimited resources and  $R$ , the **carrying capacity**, is the maximum size of the population that is sustainable by the available resources. Notice that if  $y(t)$  gets close to  $R$ , then  $y'(t)$  must get close to 0, which means that the population size can not exceed  $R$  (provided that the population size is initially between 0 and  $R$ ). One could also allow  $R$  to depend on time, but that will not be considered in this project.

Finally, we can incorporate harvesting into a population model. Suppose that the **rate of harvest** is given by  $h(t)$ , which we assume to be non-negative. Then

$$y'(t) = k(t)y(t) - h(t)$$

is a Malthusian model with time dependent harvesting and birthrate. The function  $h(t)$  specifies the instantaneous rate at which members of the population are being harvested; it is the derivative of the actual size of the harvest.

## Problem Set

1. Food Corp. has been monitoring a population of The Fish for the past three years. Below is a table with the monthly estimates for the size of the population

	Jan.	Feb.	Mar.	Apr.	May	Jun.	Jul.	Aug.	Sep.	Oct.	Nov.	Dec.
<b>Year 2003</b>	3831	3515	3329	3392	3740	4304	4873	5295	5540	5657	5682	5649
<b>Year 2004</b>	5570	5474	5412	5430	5544	5691	5816	5892	5934	5951	5956	5950
<b>Year 2005</b>	5938	5922	5913	5918	5934	5956	5974	5984	5990	5993	5994	5993

Assume that The Fish population follows a logistic model with a time dependent birthrate with a yearly period.

- (a) Estimate the carrying capacity for The Fish population's environment.
  - (b) Estimate the monthly effective birthrates; i.e.  $k(t)$ . (Hint:  $y'(t) \approx \frac{y(t+\Delta t)-y(t)}{\Delta t}$ .)
  - (c) Find a continuous function that approximately matches the monthly birthrates you found in (b). Your function should have a period of 1 year (12 months). (Hint: look for a  $k(t)$  with the form  $a + b\sin(ct + d)$ , where  $a, b, c$  and  $d$  are constants.)
  - (d) Using your estimates for the carrying capacity and birthrate function, find a solution to the logistic model. (Hint: use partial fractions.)
  - (e) Compare your solution to part (d) with the data in the table.
2. Suppose that The Fish are in an environment with unlimited resources, so the population follows a Malthusian model.

- (a) Find a solution to the model using your estimate for the birthrate function found in 1(c).
- (b) Find a solution to the model

$$y'(t) = \bar{k}y(t),$$

where  $\bar{k}$  is the average birthrate for The Fish for 1 year.

- (c) Compare your solution from (a) to your solution from (b).
3. Food Corp. wants to farm The Fish. Assume that The Fish farm provides unlimited resources, so the population will follow a Malthusian model with harvesting and a time dependent birthrate. Food Corp. would like to begin farming at the beginning of January and to maximize the number fish harvested each year without ever having to restock the farm. Assume that, without harvesting, The Fish population can be approximated using the model in 2(b). What is the maximum number of fish that can be harvested each year, if the initial population is 40000 fish?