

**1:**

(a) First,

$$e^{-x^2} = 1 - x^2 + x^4/2! - x^6/3! \pm \dots = \sum_{k=0}^{\infty} (-1)^k x^{2k}/k!$$

so:

$$\int_0^1 e^{-x^2} = \int_0^1 (1 - x^2 + x^4/2! - x^6/3! \pm \dots) dx = \int_0^1 \sum_{k=0}^{\infty} (-1)^k x^{2k}/k! dx = \sum_{k=0}^{\infty} (-1)^k x^{2k+1}/(k!(2k+1)) \Big|_0^1 = \sum_{k=0}^{\infty} (-1)^k/(k!(2k+1)) = 1 - 1/3 + 1/(2!5) - 1/(3!7) \pm \dots$$

(b) The series  $1 - 1/3 + 1/(2!5) - 1/(3!7) \pm \dots$  is an alternating series whose terms decrease in absolute value and have limit 0. Thus the sum of the series is between the sum of any  $n$  terms and the sum of  $n + 1$  terms. I.e.,

$$0.742 = 1 - 1/3 + 1/(2!5) - 1/(3!7) \leq \sum_{k=0}^{\infty} (-1)^k/(k!(2k+1)) \leq 1 - 1/3 + 1/(2!5) = 0.766 \dots$$

so the sum of the series is  $0.749 \pm .07$ .

(c) First,

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k x^{2k}/(2k)!$$

so

$$\sum_{k=1}^{\infty} (-1)^k x^{2k}/(2k)! = \cos(x) - 1$$

so

$$\sum_{k=1}^{\infty} (-1)^k \pi^{2k}/(2k)! = \cos(\pi) - 1 = -2.$$

**2:** Use the ratio test: the limit as  $k$  goes to  $\infty$  of

$$\left| \frac{\frac{x^{k+1}}{k+2}}{\frac{x^k}{k+1}} \right|$$

is  $|x|$ , so the series converges for  $|x| < 1$  and diverges for  $|x| > 1$ . Thus the radius of convergence is 1. When  $x = 1$ , we get the harmonic series, which diverges. When  $x = -1$ , we get an alternating series, whose terms decrease in absolute value with limit 0, hence is convergent. Thus the interval of convergence is  $-1 \leq x < 1$ .

**3:**

(a) First,

$$f(x) = xe^{-x^2} = x(1 - x^2 + x^4/2! - x^6/3! \pm \dots) = x \sum_{k=0}^{\infty} (-1)^k x^{2k}/k! = \sum_{k=0}^{\infty} (-1)^k x^{2k+1}/k!$$

but

$$f(x) = \sum_{k=0}^{\infty} f^{(k)}(0)x^k/k!$$

so the coefficient of the term in which  $x$  has the exponent 100 is

$$f^{(100)}(0)x^{100}/100!$$

Since only odd powers of  $x$  actually appear, this means

$$f^{(100)}(0)x^{100}/100! = 0$$

hence that  $f^{(100)}(0) = 0$ .

(b) First,  $g(x) = (1+x)^{-1/2}$ ,  $g'(x) = (-1/2)(1+x)^{-3/2}$ ,  $g''(x) = (-1/2)(-3/2)(1+x)^{-5/2}$ , and  $g'''(x) = (-1/2)(-3/2)(-5/2)(1+x)^{-7/2}$ . So, since we want a Taylor polynomial centered at  $x = 0$ , we have

$$P_3(x) = g(0) + g'(0)x/1! + g''(0)x^2/2! + g'''(0)x^3/3!$$

or

$$P_3(x) = 1 - x/2 + 3x^2/8 - 5x^3/16$$

**4:**

(a)  $\sum_{k=0}^{\infty} (-1)^k/\sqrt{k+1}$  converges conditionally: it does not converge absolutely (since the absolute values of the terms decrease and  $1/\sqrt{k+1}$  is continuous, we can apply the integral test, but  $\int_0^{\infty} dx/\sqrt{x+1}$  diverges, hence so does the series of absolute values), but it does converge (since it is an alternating series where the absolute values of the terms decrease with limit 0, it converges by the alternating test).

(b)  $\sum_{k=2}^{\infty} (-1)^k/(k(\ln k)^2)$  converges absolutely (since the absolute values of the terms are decreasing and  $1/(k(\ln k)^2)$  is continuous for  $k \geq 2$ , we can apply the integral test, but  $\int_2^{\infty} dx/(x(\ln x)^2) = 1/\ln 2$  converges, hence so does the series of absolute values, hence the series converges absolutely).

(c) By comparison with the harmonic series,

$$\sum_{k=1}^{\infty} k/(2k^2 - 1) \geq \sum_{k=1}^{\infty} k/(2k^2) = (1/2) \sum_{k=1}^{\infty} 1/k$$

the given series diverges.

**5:**

(a) The curve is a segment of the line  $y = (1/2)x + 2$ , so there is no place where the slope of the tangent line is 1.

(b) Using the formula, we have

$$\int_0^1 2\pi(2 - 2t^2)\sqrt{(4t)^2 + (2t)^2} dt$$

**6:**

(a)  $(x, y) = (-2, 4) + t(6 - (-2), 4 - 1)$  for  $0 \leq t \leq 1$ , or  $x(t) = -2 + 8t$  and  $y(t) = 4 + 3t$  for  $0 \leq t \leq 1$ .

(b)  $(x, y) = (2, 1) + 2(\cos(-t), \sin(-t))$  or  $x(t) = 2 + 2\cos(t)$  and  $y(t) = 1 - 2\sin(t)$ .

**Bonus:** Colin Maclaurin was from Scotland; he discovered the integral test.