

M107 Project: Brief Solutions

This is not meant as a model to follow in writing a complete report. Rather, it is just a sketch of how to work out the individual items mathematically that were involved in this project.

Recall that the centroid of a line segment is just its midpoint. For a plane figures positioned so that each of its points is between $x = a_1$ and $x = b_1$, and between $y = a_2$ and $y = b_2$, the coordinates of the centroid are (r_1, r_2) where

$$r_1 = (1/A) \int_{a_1}^{b_1} x L_1(x) dx \qquad r_2 = (1/A) \int_{a_2}^{b_2} y L_2(y) dy,$$

A is the area of the figure, $L_1(x)$ is the total the length of the vertical slice through the figure at a given x value, $L_2(y)$ is the total the length of the horizontal slice through the figure at a given y value. For a space figure, every point (x, y, z) of which lies in the ranges $a_1 \leq x \leq b_1$, $a_2 \leq y \leq b_2$, $a_3 \leq z \leq b_3$, the coordinates of the centroid are (r_1, r_2, r_3) , where

$$r_1 = (1/V) \int_{a_1}^{b_1} x A_1(x) dx \qquad r_2 = (1/V) \int_{a_2}^{b_2} y A_2(y) dy \qquad r_3 = (1/V) \int_{a_3}^{b_3} z A_3(z) dz,$$

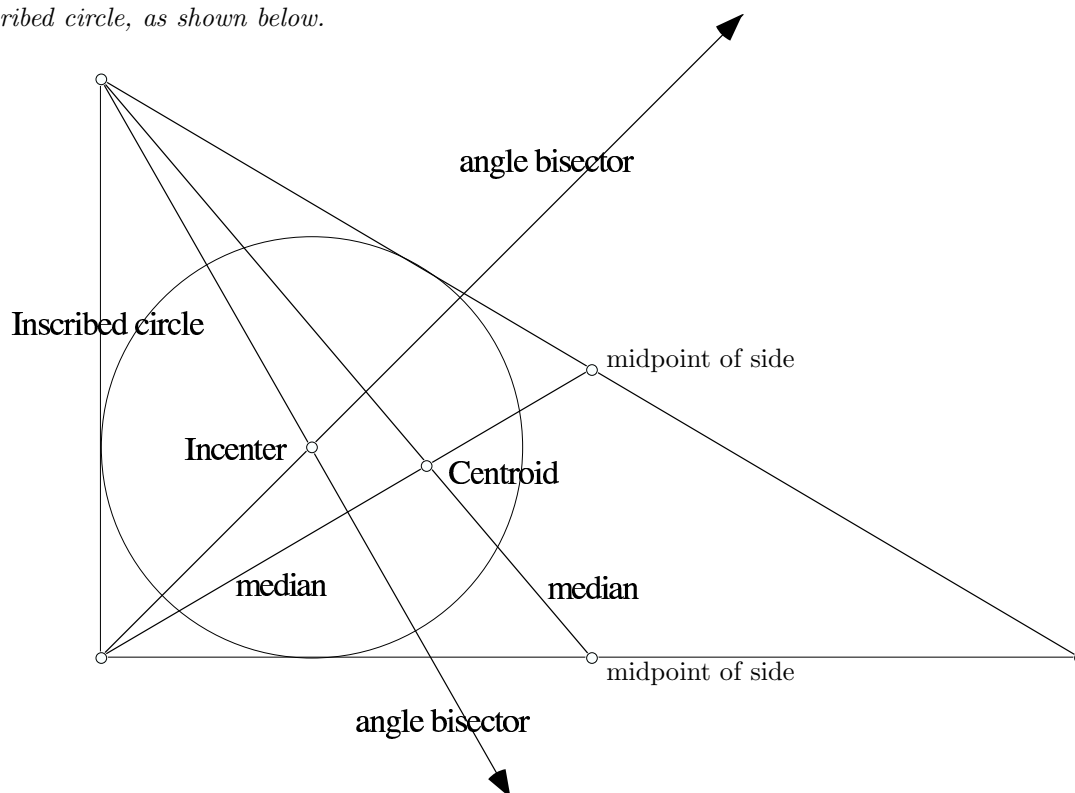
where $A_1(x)$ denotes the area of the region obtained by intersecting the figure with the plane perpendicular to the x -axis at the given x -value, $A_2(y)$ denotes the area of the region obtained by intersecting the figure with the plane perpendicular to the y -axis at the given y -value, $A_3(z)$ denotes the area of the region obtained by intersecting the figure with the plane perpendicular to the z -axis at the given z -value, and V is the volume of the figure.

- (1) Consider the triangular figure whose vertices are $(0,0)$, $(8,0)$ and $(0,10)$. Cut it out of cardboard (perhaps use the cardboard at the back of a pad of paper). Use the physical method described above, putting the pins as close as possible to the vertices of the triangle, to get three lines that intersect at the centroid. Also draw on this triangle the angle bisectors, and include this cardboard cutout in the write up you turn in.

Alternatively, you can hold the triangle face up and try to find the spot where you can put your finger underneath so that it balances.

- (2) An elementary school math textbook from a well known publisher (used by some schools in Omaha) claims that the centroid is where the angle bisectors intersect. Is this true? Why or why not?

The centroid is where the medians intersect; the angle bisectors intersect at the incenter, the center of the inscribed circle, as shown below.

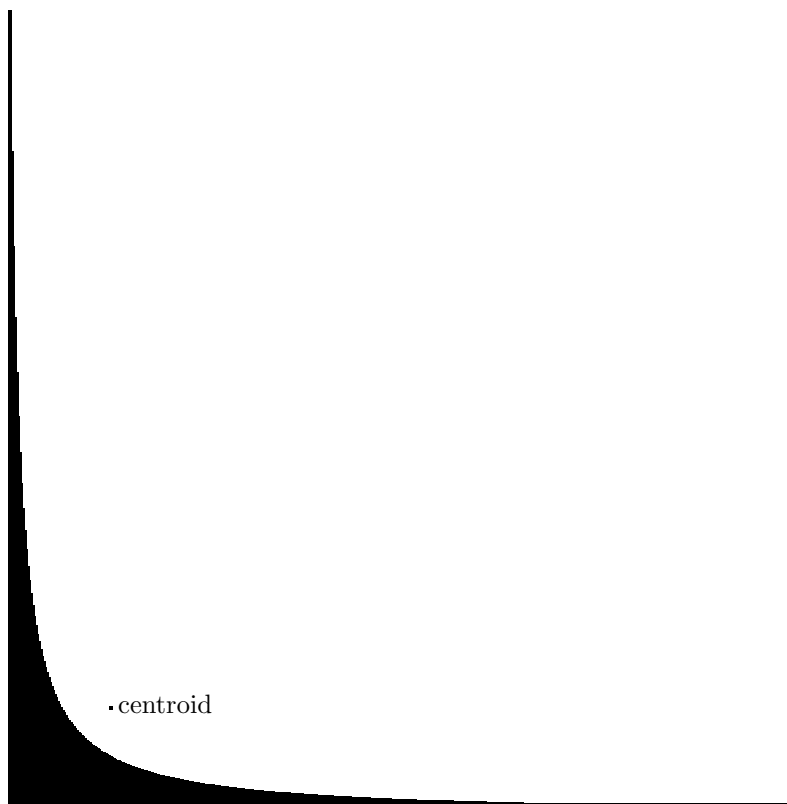


- (3) Let a and b be positive numbers. Consider the triangle whose vertices are $(0,0)$, $(a,0)$ and $(0,b)$. Use the integral formulas to find the coordinates of the centroid (your answer will be in terms of a and b). Show what your integrals were. Compare your mathematical answer to what you found experimentally in part (1), and plot the centroid using your mathematical answer on your cardboard cutout.

The area is $ab/2$. The hypotenuse of the triangle is given by $y = -bx/a + b$. The moment in the x -direction is $\int_0^a x(-bx/a + b) dx = (-bx^3/(3a) + bx^2/2)|_0^a = -ba^2/3 + ba^2/2 = ba^2/6$. Thus the x -coordinate of the centroid is $(ba^2/6)/(ab/2) = a/3$. The y -coordinate similarly is $b/3$.

- (4) Now consider the plane figure bounded on the left by $x = 1/10$, above by $y = x^{-1}$, and below by $y = 1/10$. Find the coordinates of the centroid exactly; show how you used the integral formulas to find the centroid. Draw a graph of the figure, and plot the centroid.

The area of the figure is $\int_{0.1}^{10} (x^{-1} - 0.1) dx = (\ln(x) - x/10)|_{0.1}^{10} = 2\ln(10) - 0.99$ or about 3.61517. The moment in the x -direction is $\int_{0.1}^{10} (1 - x/10) dx = (x - x^2/20)|_{0.1}^{10} = 9.9 - 5 + 1/2000 = 4.9005$. The x -coordinate of the centroid is thus $4.9005/3.61517$ or about 1.355538. By symmetry, this is also the y -coordinate. See the figure below; note that the centroid is not inside the figure.



- (5) Find the coordinates of the centroid exactly for the figure bounded above by $y = e^x - 1$, below by $y = 0$ and on the right by $x = 2$. Show how you do it and how you work out the integrals you get.

The area is $\int_0^2 e^x - 1 \, dx = (e^x - x)|_0^2 = e^2 - 3$, or about 4.389. The x -moment is $\int_0^2 x(e^x - 1) \, dx = (xe^x - e^x - x^2/2)|_0^2 = e^2 - 1$, or about 6.389. Thus the x -coordinate of the centroid is $(e^2 - 1)/(e^2 - 3)$, or about 1.46. To find the y -coordinate, we need to express y in terms of x : if $y = e^x - 1$, then $\ln(y + 1) = x$. The width of the figure as a function of y is thus $2 - \ln(y + 1)$, so the y -moment is $\int_0^{e^2-1} y(2 - \ln(y + 1)) \, dy = (y^2 - [(1/2)(y + 1)^2 \ln(y + 1) - (y + 1)\ln(y + 1) - (y + 1)^2/4 + (y + 1)])|_0^{e^2-1} = (e^2 - 1)^2 - ((3/4)(e^4 - 1) - e^2) = (e^4 - 4e^2 + 7)/4$, or about 8.01. Thus the y -coordinate of the centroid is $((e^4 - 4e^2 + 7)/4)/(e^2 - 3)$, or about 1.825.

- (6) Find the coordinates of the centroid exactly for the figure bounded above by $y = \sin(x)$ and below by $y = 0$ for $0 \leq x \leq \pi/2$. Show how you do it and how you work out the integrals you get.

The area is $\int_0^{\pi/2} \sin(x) \, dx = -\cos(x)|_0^{\pi/2} = 1$. The x -moment is $\int_0^{\pi/2} x \sin(x) \, dx = (\sin(x) - x \cos(x))|_0^{\pi/2} = 1$. The y -moment is $\int_0^1 y(\pi/2 - \arcsin(y)) \, dy = \pi y^2/4|_0^1 - \int_0^1 (y \arcsin(y)) \, dy = \pi/4 - \int_0^1 (y \arcsin(y)) \, dy$. Integrating by parts gives

$$\int_0^1 (y \arcsin(y)) \, dy = (1/2)y^2 \arcsin(y)|_0^1 - (1/2) \int_0^1 y^2 / \sqrt{1 - y^2} \, dy = \pi/4 - (1/2) \int_0^1 y^2 / \sqrt{1 - y^2} \, dy.$$

Thus $\int_0^1 y(\pi/2 - \arcsin(y)) dy = (1/2) \int_0^1 y^2/\sqrt{1-y^2} dy$. Now substitute $y = \sin(\theta)$ to get

$$\int_0^1 y^2/\sqrt{1-y^2} dy = \int_0^{\pi/2} \sin^2(\theta) d\theta,$$

which, using the half-angle formula, for example, is $\pi/4$. Thus the y -coordinate is $\pi/8$, or about 0.393.

- (7) For this one, just find the x -coordinate of the centroid (but do it exactly), for the figure bounded on the left by $x = 1$, above by $y = (x^3 + x)^{-1}$, below by $y = 0$ and on the right by $x = a$. What is the limit of this x -coordinate as $a \rightarrow \infty$? Is there a maximum position to the right that you would never need to go beyond in placing a fulcrum on which this figure could balance? Relate this to improper integrals.

The area is $\int_1^a dx/(x^3 + x) = \int_1^a dx/(x(x^2 + 1)) = \int_1^a (x^{-1} - x/(x^2 + 1)) dx = [\ln(x) - (1/2)\ln(x^2 + 1)]_1^a = \ln(a) - (1/2)\ln(a^2 + 1) + (1/2)\ln(2)$. The x -moment is $\int_1^a (1 - x^2/(x^2 + 1)) dx = \int_0^a (x^2 + 1)^{-1} dx = \arctan(x)|_1^a = \arctan(a) - \pi/4$. Thus the x -coordinate of the centroid is $(\arctan(a) - \pi/4)/(\ln(a) - (1/2)\ln(a^2 + 1) + (1/2)\ln(2))$. Now $\lim_{a \rightarrow \infty} [(\arctan(a) - \pi/4)/(\ln(a) - (1/2)\ln(a^2 + 1) + (1/2)\ln(2))] = [\lim_{a \rightarrow \infty} (\arctan(a) - \pi/4)]/[\lim_{a \rightarrow \infty} (\ln(a) - (1/2)\ln(a^2 + 1) + (1/2)\ln(2))] = (\pi/2 - \pi/4)/((1/2)\ln(2)) = \pi/(2\ln(2))$, or about 2.266. Thus the fulcrum never needs to be put further to the right than 2.266. If we consider the region bounded by $x = 1$, $y = (x^3 + x)^{-1}$ and $y = 0$, and extending infinitely far off to the right, the x -coordinate of its centroid is given by $(\int_1^\infty \frac{x}{x^3+x} dx)/(\int_1^\infty \frac{1}{x^3+x} dx)$.

- (8) Write down a formula for the coordinates of the centroid of a line segment on the x -axis from the origin to $x = a$. Next write down a formula for the coordinates of the centroid of a triangle whose coordinates are at $(0, 0)$, $(a, 0)$, and $(0, b)$, where a and b are positive numbers. Then predict what the coordinates of the centroid should be for the tetrahedron with vertices at $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ (a tetrahedron is a pyramid with a triangular base), where a , b and c are positive numbers. Verify your answer using integration.

The centroid of the line segment is its midpoint, $(a/2, 0)$. The centroid of the triangle is $(a/3, b/3)$. It is reasonable to predict that the centroid of the tetrahedron is $(a/4, b/4, c/4)$. From high school geometry we may recall that the volume of a pyramid is one third the area of the base times the height. Thus the volume of the tetrahedron is $abc/6$. We can find this by calculus, without using multiple integrals. Use the slicing method (see p. 413, and Problem #11, p. 423, where you find the volume of the magnificent pyramid at Gizeh, using horizontal slices). We will take slices perpendicular to the x -axis. The slices are triangles, as shown in the figure below. The equation of the line through the top vertex of each triangular slice is $z = -cx/a + c$, while the equation of the line through the right vertex of each triangular slice is $y = -bx/a + b$. Thus each slice will be a triangle of height $h = -cx/a + c$ and base $b = -bx/a + b$, making the area of the slice $A_1(x) = bh/2 = (-bx/a + b)(-cx/a + c)/2$. Multiplying this out and integrating, we find the volume to be $\int_0^a A_1(x) dx = abc/6$.

The x -moment of the tetrahedron is now $\int_0^a xA_1(x) dx = a^2bc/24$, so the x -coordinate of the centroid is $(a^2bc/24)/(abc/6) = a/4$.

Do the area functions $A_2(y)$ and $A_3(z)$ similarly, but use slices perpendicular to the y -axis to get $A_2(y)$, and use horizontal slices to get $A_3(z)$. What you'll find is that $A_2(y) = (-cy/b + c)(-ay/b + a)/2$, $A_3(z) = (-bz/c + b)(-az/c + a)/2$, the y -moment is $b/4$ and the z -moment is $c/4$, as predicted.

