

M107 Project Instructions

Due Date: Tuesday, November 23, 2004

Guidelines: This project is a group project. You may need to use various references, but be sure to give a proper citation for any information that you include in your project write up that is not original to you. Also, your grade will be based partly on the quality of your written work. Use complete sentences, good grammar, correct spelling and correct punctuation. The paper you turn in should be a mix of equations, formulas and prose. Illustrations and tables should be clearly labeled. You should write your answers in such a way that it can be read and understood by anyone who knows the material for this course. This is your target audience, not the instructor. Finally, neatness counts, so the project should be neatly typed or written on good paper (not torn from a notebook). Your write up should not simply be numbered answers to the numbered items listed below; they are numbered just to make it easier for you to see what your project write up must cover. Instead, your write up should read as a coherent whole, an essay, into which the answers to the items listed are incorporated.

About Group Projects: To get everyone involved and the group functioning smoothly, it is a good idea to meet as early as possible to arrange meeting times, etc. It might be helpful to bear in mind that there are at least four roles to be played by various participants at various times: the chair, reporter, scheduler and scribe. The role of the chair is to try to get everyone involved and make sure everyone is understanding the ideas developed by the group. The reporter jots down the ideas of the group as they are discussed. The scheduler finds times and places where everyone in the group can meet, and finally, the scribe writes up the final report for the group. These jobs can be rotated on a per meeting basis if the group wishes. However, everyone should proofread the final draft and help in the other duties as they see fit.

Participation: When the project is turned in, students may be asked to evaluate the level of participation by other group members by way of a project participation report to be filled out by each member individually and turned in to the recitation instructor.

Grading: Projects count 40 points. Points will be assigned as follows:

- Completeness: 10
- Correctness: 10
- Grammar: 4
- Full sentences: 4
- Spelling: 4
- Neatness: 4
- Originality: 4

The Project

General Background: The notion of centroid or, more generally, the center of mass (also known as center of gravity), turns up everywhere in the real world, from sports (what makes the Fosbury flop work so well?) to car design (why do SUVs tip over so easily?). This project is intended to help develop your physical notion of center of mass, and explain how it can be computed mathematically. (Another aim is to give you practice using methods of integration!)

First consider a long weighted rod (e.g., a pole slung with donut shaped weights in fixed positions). Imagine placing a fulcrum under the pole with the pole positioned horizontally so that the pole could teeter-totter on the fulcrum. If the fulcrum is positioned at just the right point, the pole would balance. That point is the center of mass of the weighted pole.

Now consider a plane figure (a shape cut out of cardboard, say). To find the center of mass, poke a pin somewhere through the figure so that the figure can rotate freely about the pin. Unless you happen to poke the pin through the center of mass itself, the figure will rotate until the center of mass hangs directly below the pin. (Thus it's best if you poke the pin as far from what looks like the center of the object as possible, so that the figure will be pretty unbalanced as it swings around the pin.)

If you let the figure swing around the pin until it comes to a stop hanging under the pin, and then draw a vertical line from the pin straight down, the line will pass through the center of mass. (To be sure the line is plumb, that is, vertical, you could use a plumb bob: attach a weight to a string, and attach the other end of the string to the pin. The string will then be a plumb line, showing the vertical line through the pin position on the figure.) If you now repeat this for another position of the pin, you will get a second line; the point where the two lines intersect is the center of mass. Doing this a third time will check your accuracy, since the third line should also go through the same point as do the first two lines. (This method also works to determine the center of mass of a three dimensional object: hang the object from two different points. From each point, find the vertical line through the point. The two lines intersect at the center of mass. Unfortunately, these lines can meet inside the object, so it's not as easy to actually find where the lines meet as it is for a plane figure; for the plane figure you can just use a pencil to draw in the lines determined by the plumb lines. It is also worth mentioning that the center of mass is not always inside an object, even for a plane figure. For example, an L-shaped figure's center of mass is somewhere in the space between the two legs of the L. This is an essential feature of the Fosbury flop.)

The mathematical method for finding the center of mass of an object whose density is not the same everywhere involves a double or triple integral (i.e., M208 level material), so we'll just consider objects of constant density (such as a solid pyramid made out of a single material, or a cardboard cut-out, versus something like an automobile, which is made out of various materials of various densities). For figures of constant density, the center of mass is also called the centroid.

Mathematical Background: Here's how, mathematically, to find the coordinates of the centroid of a figure of constant density. (When the density is constant, it factors out of the formulas, so the specific value of the density doesn't matter.)

- Line segments: The centroid of a line segment is just its midpoint.
- Plane figures: Say a plane figure is positioned so that each of its points is between $x = a_1$ and $x = b_1$, and between $y = a_2$ and $y = b_2$. Consider the vertical line through a given x value. It will intersect the figure in one or more line segments, depending on the shape and position of the figure. (For example, if the figure is a washer with a hole, lying flat in the plane, sometimes the vertical line will intersect it in a single line segment, sometimes in two.) Let $L_1(x)$ be the sum of the lengths of these line segments. Similarly, let $L_2(y)$ be the sum of the lengths of the line segments at which the figure intersects the line perpendicular to the y -axis through a given y value. Let A be the area of the figure. The coordinates of the centroid of the figure are (r_1, r_2) , where:

$$r_1 = (1/A) \int_{a_1}^{b_1} xL_1(x) dx \qquad r_2 = (1/A) \int_{a_2}^{b_2} yL_2(y) dy$$

- Space figures: Consider a three dimensional figure, every point (x, y, z) of which lies in the ranges $a_1 \leq x \leq b_1$, $a_2 \leq y \leq b_2$, $a_3 \leq z \leq b_3$. Let $A_1(x)$ denote the *area* of the region obtained by intersecting the figure with the plane perpendicular to the x -axis at the given x -value. Similarly define $A_2(y)$ (using planes perpendicular to the y -axis) and $A_3(z)$ (using planes perpendicular to the z -axis). Let V be the volume of the figure. Then the coordinates of the centroid are (r_1, r_2, r_3) , where

$$r_1 = (1/V) \int_{a_1}^{b_1} xA_1(x) dx \qquad r_2 = (1/V) \int_{a_2}^{b_2} yA_2(y) dy \qquad r_3 = (1/V) \int_{a_3}^{b_3} zA_3(z) dz$$

Here are the specific items your project write up must include:

- (1) Consider the triangular figure whose vertices are $(0, 0)$, $(8, 0)$ and $(0, 10)$. Cut it out of cardboard (perhaps use the cardboard at the back of a pad of paper). Use the physical method described above, putting the pins as close as possible to the vertices of the triangle, to get three lines that intersect at the centroid. Also draw on this triangle the angle bisectors, and include this cardboard cutout in the write up you turn in.

- (2) An elementary school math textbook from a well known publisher (used by some schools in Omaha) claims that the centroid is where the angle bisectors intersect. Is this true? Why or why not?
- (3) Let a and b be positive numbers. Consider the triangle whose vertices are $(0, 0)$, $(a, 0)$ and $(0, b)$. Use the integral formulas to find the coordinates of the centroid (your answer will be in terms of a and b). Show what your integrals were. Compare your mathematical answer to what you found experimentally in part (1), and plot the centroid using your mathematical answer on your cardboard cutout.
- (4) Now consider the plane figure bounded on the left by $x = 1/10$, above by $y = x^{-1}$, and below by $y = 1/10$. Find the coordinates of the centroid exactly; show how you used the integral formulas to find the centroid. Draw a graph of the figure, and plot the centroid.
- (5) Find the coordinates of the centroid exactly for the figure bounded above by $y = e^x - 1$, below by $y = 0$ and on the right by $x = 2$. Show how you do it and how you work out the integrals you get.
- (6) Find the coordinates of the centroid exactly for the figure bounded above by $y = \sin(x)$ and below by $y = 0$ for $0 \leq x \leq \pi/2$. Show how you do it and how you work out the integrals you get.
- (7) For this one, just find the x -coordinate of the centroid (but do it exactly), for the figure bounded on the left by $x = 1$, above by $y = (x^3 + x)^{-1}$, below by $y = 0$ and on the right by $x = a$. What is the limit of this x -coordinate as $a \rightarrow \infty$? Is there a maximum position to the right that you would never need to go beyond in placing a fulcrum on which this figure could balance? Relate this to improper integrals.
- (8) Write down a formula for the coordinates of the centroid of a line segment on the x -axis from the origin to $x = a$. Next write down a formula for the coordinates of a triangle whose coordinates are at $(0, 0)$, $(a, 0)$, and $(0, b)$, where a and b are positive numbers. Then predict what the coordinates should be for the tetrahedron with vertices at $(0, 0, 0)$, $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ (a tetrahedron is a pyramid with a triangular base), where a , b and c are positive numbers. Verify your answer using integration.