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Rational Surfaces with $K^2 > 0$

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The main but not all of the results in this paper concern rational surfaces X for which the self-intersection K_X^2 of the anticanonical class $-K_X$ is positive. In particular, it is shown that no superabundant numerically effective divisor classes occur on any smooth rational projective surface X with $K_X^2 > 0$. As an application, it follows that any 8 or fewer (possibly infinitely near) points in the projective plane $\mathbf{P}^2$ are in good position. This is not true for 9 points, and a characterization of the good position locus in this case is also given. Moreover, these results are put into the context of conjectures for generic blowings up of $\mathbf{P}^2$. All results are proven over an algebraically closed field of arbitrary characteristic.

I. Introduction

If $\mathcal{F}$ is a divisor class on the blowing up X of $\mathbf{P}^2$ at $n \leq 8$ general points $p_1, \ldots, p_n \in \mathbf{P}^2$, then there is a straightforward algorithm [H1] for computing the dimension $h^0(X, \mathcal{F})$ of the space of sections of $\mathcal{F}$, assuming the coefficients expressing $\mathcal{F}$ as a linear combination of the class $\mathcal{E}_0$ of a line and the classes $\mathcal{E}_i$, $1 \leq i \leq n$, of the exceptional loci are given.

The algorithm depends on the following well-known theorem (for convenience, we will give a proof in Section II). Note that EFF stands for the monoid in $\mathrm{Pic}(X)$ of classes of effective divisors, and NEF stands for the cone of numerically effective classes (those classes whose intersection with every effective class is nonnegative).

Theorem I.1: *Let X be a blowing up of $\mathbf{P}^2$ at $n \leq 8$ general points.*

(a) *Then $\mathcal{F} \in$ NEF if and only if $\mathcal{F}^2 \geq 0$, $\mathcal{F} \cdot \mathcal{E}_0 \geq 0$ and $\mathcal{F} \cdot \mathcal{E} \geq 0$ whenever $\mathcal{E}$ is the class of an exceptional curve (i.e., a smooth rational curve of self-intersection -1). Moreover, a class is effective if and only if it is a nonnegative linear combination of a numerically effective class $\mathcal{F}$ with $\mathcal{F}^2 - K_X \cdot \mathcal{F} \geq 0$ and classes of exceptional curves.*

(b) *If $\mathcal{F} \in$ NEF, then $h^2(X, \mathcal{F}) = 0$; if also $\mathcal{F} \in$ EFF, then $h^1(X, \mathcal{F}) = 0$.*

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More generally, whenever X is a blowing up of $\mathbf{P}^2$ for which Theorem I.1(a) and Theorem I.1(b) hold, the algorithm mentioned above provides a means of determining h^0 for any class $\mathcal{F}$ on X. Thus it is of interest to know when properties (a) and (b) hold. Conjecturally, (a) and (b) hold whenever X is a blowing up of $\mathbf{P}^2$ at "sufficiently general" points (see [Gi], [Hi], [H3], [H5] for equivalent conjectures). [For $n > 8$, note that choosing n sufficiently general points may involve extending the ground field, k. If X is the blowing up of 9 general points of $\mathbf{P}^2$ over the algebraic closure k of a finite field, then $-K_X$ is the class of a smooth cubic D, and the restriction of $-K_X$ to D is a torsion class (of order r, say), with the result that $-rK_X$ is effective, numerically effective and superabundant, and thus X fails (b).]

It seems worthwhile to consider properties (a) and (b) separately. Property (a) clearly depends on a genericity hypothesis; a choice of sufficiently many points on a given plane curve C gives rise to a class (in particular, the class $\mathcal{C}$ of the proper transform of C) on the blow up of the points which is not in the span of NEF and exceptional curves. (In fact it can be shown that the statement in (a) regarding NEF implies that X supports no reduced, irreducible curves of negative self-intersection but exceptional curves, whereas if C has degree d and one blows up $d^2 + 2$ points on C, then $\mathcal{C}^2 < -1$.) However, it turns out that (b) can hold even for quite special choices of points.

Our main interest in this paper is property (b); when X is obtained by blowing up points $p_1, \ldots, p_n$ of $\mathbf{P}^2$ (possibly infinitely near), we say that the points are in *good position* if (b) holds for X. Our main result is that (b) holds for all rational surfaces X in the case that $K_X^2 > 0$, and so, as a corollary, any eight or fewer points of $\mathbf{P}^2$ (possibly infinitely near) are in good position. After recalling or establishing the needed background in Section II, we prove our main result in Section III, and end with the corollary and a few remarks.

II. Preliminary Results

By *surface* we will always mean a smooth projective surface (usually rational) over an algebraically closed field of arbitrary characteristic. Also, we will say that a divisor class is *effective* if it is the class of an effective divisor. Moreover, when convenient, given a curve N on a surface X and a divisor class $\mathcal{F}$ on X, we will write $h^i(N, \mathcal{F})$ for the dimension of the i-th cohomology of the restriction $\mathcal{F} \otimes \mathcal{O}_N$ of $\mathcal{F}$ to N. We begin by recalling Riemann-Roch for rational surfaces.

Lemma II.1: *Let X be a smooth projective rational surface, and let $\mathcal{F}$ be a divisor class on X.*

(a) We have: $h^0(X, \mathcal{F}) - h^1(X, \mathcal{F}) + h^2(X, \mathcal{F}) = (\mathcal{F}^2 - K_X \cdot \mathcal{F})/2 + 1.$

(b) If $\mathcal{F}$ is effective, then $h^2(X, \mathcal{F}) = 0$.

Proof: (a) This is just the Riemann-Roch formula in the case of a rational surface.

(b) Since $\mathcal{F}$ is effective, we have an injection $K_X - \mathcal{F} \to K_X$ of sheaves, hence an injection on global sections. But $h^2(X, \mathcal{O}_X) = 0$ since X is rational, so by duality K_X has no nontrivial global sections. Hence neither does $K_X - \mathcal{F}$, so again by duality $h^2(X, \mathcal{F}) = h^0(X, K_X - \mathcal{F})$. $\diamondsuit$

We recall some standard facts about pullbacks:

Lemma II.2: *Let $\pi : Y \to X$ be a birational morphism of smooth projective rational surfaces, $\pi^* : Pic(X) \to Pic(Y)$ the corresponding homomorphism on Picard groups, and let $\mathcal{L}$ be a divisor class on X.*

(a) The map π^* is an injective intersection-form preserving map of free abelian groups of finite rank.

(b) The map π^* preserves dimensions of cohomology groups; i.e., $h^i(X, \mathcal{L}) = h^i(Y, \pi^*\mathcal{L})$ for every i.

(c) The map π^* preserves effectivity; i.e., $\mathcal{L}$ is effective if and only if $\pi^*\mathcal{L}$ is.

(d) The map π^* preserves numerical effectivity; i.e., $\mathcal{L} \cdot F \geq 0$ for every effective divisor F on X if and only if $(\pi^*\mathcal{L}) \cdot F' \geq 0$ for every effective divisor F' on Y.

Proof: (a) See [Ha, V].

(b) This follows from [Ha, V.3.4] and the Leray spectral sequence.

(c) This follows from (b) with $i = 0$.

(d) If $\pi^*\mathcal{L}$ is numerically effective, then numerical effectivity for $\mathcal{L}$ follows from (a) and (c). Conversely, suppose $\mathcal{L}$ is numerically effective. Factor $\pi : Y \to X$ into a sequence $Y = X_n \to \cdots \to X_0 = X$ of morphisms $\pi_i : X_{i+1} \to X_i$, where π_i is the blowing up of a point $p_i \in X_i$. By induction, it suffices to consider the case that $n = 1$, which is precisely [H2, Lemma 1.4]. $\diamondsuit$

Proposition II.3: *Let $\mathcal{F}$ be a numerically effective class on a smooth projective rational surface X. Then $h^2(X, \mathcal{F}) = 0$ and $\mathcal{F}^2 \geq 0$.*

Proof: We first reduce to the case that X has a birational morphism to $\mathbf{P}^2$. Since X is a rational surface, it at least has a birational transformation to $\mathbf{P}^2$. Thus, as in the proof of [Ha, V.5.5], there is a smooth projective rational surface Y with birational morphisms to both X and $\mathbf{P}^2$. Let π denote the morphism to X. Now, $\pi^*\mathcal{F}$ is numerically effective by Lemma II.2(d), $(\pi^*\mathcal{F})^2 = \mathcal{F}^2$ follows from Lemma II.2(a), and so $\pi^*\mathcal{F} \cdot K_Y = \mathcal{F} \cdot K_X$ follows from Lemma II.2(a), (b) and Lemma II.1(a). Thus it suffices to show that $(\pi^*\mathcal{F})^2 \geq 0$. I.e., we may assume that there is a birational morphism $X \to \mathbf{P}^2$.

A standard fact [Ha, V.5.3] states that this morphism factors as a sequence $X = X_n \to \cdots \to X_0 = \mathbf{P}^2$ of morphisms $\pi_i : X_{i+1} \to X_i$, where π_i is the blowing up of a point $p_i \in X_i$. Clearly, (by blowing up Y of the preceding paragraph a few more times if necessary) we may assume $n > 2$. From this factorization we obtain the classes $\mathcal{E} = \{\mathcal{E}_0, \ldots, \mathcal{E}_n\}$, where $\mathcal{E}_0$ is the pullback to X of the class of a line in $\mathbf{P}^2$, and $\mathcal{E}_i$ for $i > 0$ is the class of the total transform of p_i with respect to the morphism $X \to X_i$.

Such a set $\mathcal{E}$ of classes is called an *exceptional configuration* [H2]; it gives an orthogonal basis of Pic(X) in which $-\mathcal{E}_0^2 = \mathcal{E}_1^2 = \cdots = \mathcal{E}_n^2 = -1$. Thus we can write $\mathcal{F} = \sum_i a_i \mathcal{E}_i$ for some integers a_i. We also have orthogonal involutions $s_0, \ldots, s_{n-1}$ of Pic(X) defined via $s_i(x) = x + (x \cdot r_i) r_i$, where $r_0 = \mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2 - \mathcal{E}_3$, and $r_i = \mathcal{E}_i - \mathcal{E}_{i+1}$, for $i > 0$. (Note that, since $n > 2$, r_0 is defined.) These generate a subgroup W, called the *Weyl group*, of the orthogonal group of Pic(X). It is also useful to note that $-K_X = 3\mathcal{E}_0 - \mathcal{E}_1 - \cdots - \mathcal{E}_n$ [Ha, II.8.20.1, V.3.3], hence $K_X \cdot r_i = 0$ for all i, so $wK_X = K_X$ for all $w \in W$. By numerical effectivity, $\mathcal{F} \cdot \mathcal{E}_0 \geq 0$ so $(K_X - \mathcal{F}) \cdot \mathcal{E}_0 \leq -3$, and, since $\mathcal{E}_0$ is numerically effective (being irreducible of positive self-intersection), we see $K_X - \mathcal{F}$ is not effective, hence that $h^2(X, \mathcal{F}) = 0$, as claimed.

Now suppose that $\mathcal{F}^2 < 0$. For $l \geq 0$, $\mathcal{F} + l\mathcal{E}_0$ is numerically effective, hence $h^2(X, \mathcal{F} + l\mathcal{E}_0) = 0$, so by Lemma II.1 for l sufficiently large $\mathcal{F} + l\mathcal{E}_0$ is effective, and we have $\mathcal{F} \cdot (\mathcal{F} + l\mathcal{E}_0) \geq 0$, which implies $\mathcal{F} \cdot \mathcal{E}_0 > 0$. Hence in $\mathrm{Pic}(X) \otimes \mathbf{Q}$, for $\epsilon \in \mathbf{Q}$ with $-\mathcal{F}^2/(2\mathcal{F} \cdot \mathcal{E}_0) < \epsilon < -\mathcal{F}^2/(\mathcal{F} \cdot \mathcal{E}_0)$, we have $(\mathcal{F} + \epsilon\mathcal{E}_0)^2 > 0$ but $(\mathcal{F} + \epsilon\mathcal{E}_0) \cdot \mathcal{F} < 0$. This contradicts the fact that, for sufficiently large multiples t of the denominator of ϵ, $t(\mathcal{F} + \epsilon\mathcal{E}_0) \in \mathrm{Pic}(X)$ is effective. $\diamondsuit$

Corollary II.4: *On a smooth projective rational surface, a numerically effective divisor class meeting the canonical class nonnegatively is effective. In particular, effectivity of the anticanonical class implies effectivity of all numerically effective classes.*

Proof: This follows from Lemma II.1(a) and Proposition II.3. $\diamondsuit$

Lemma II.5: *Let X be a smooth projective rational surface, and let $\mathcal{C}$ be an effective divisor class without fixed components. If $K_X^2 > 0$, then $h^1(X, \mathcal{C}) = 0$.*

Proof: Since $K_X^2 > 0$ implies by the Hodge index theorem that the subgroup $K_X^\perp \subset \mathrm{Pic}(X)$ perpendicular to K_X is negative definite, this is Theorem 4.2 of [H5]. $\diamondsuit$

The following lemma is an adaptation of Theorem 1.7 of [A] and is the key to our results.

Lemma II.6: *Let X be a smooth projective surface supporting an effective divisor N and a divisor class $\mathcal{F}$ which meets every component of N nonnegatively. If $h^1(N, \mathcal{O}_N) = 0$, then $h^0(N, \mathcal{F}) > 0$ and $h^1(N, \mathcal{F}) = 0$.*

Proof: By [A, Theorem 1.7], every component of N is smooth and rational and $h^1(M, \mathcal{O}_M) = 0$ for every positive subcycle $M \leq N$. Write N as a sum of its components: $N = \sum_i r_i X_i$, where each r_i is a positive integer and each X_i is a smooth rational curve. The proof, mutatis mutandis, now follows the proof of Theorem 1.7 [A], by induction on $\sum_i r_i$, noting that the result holds when $\sum_i r_i = 1$.

First, there is a component, say X_0, of N with $N \cdot X_0 \leq 1 + X_0^2$. [If not, then $N \cdot X_i \geq 2 + X_i^2$ for every i. But rationality of the components and adjunction give $K_X \cdot X_i = -2 - X_i^2$. Thus $N^2 + K_X \cdot N = (N + K_X) \cdot \sum_i r_i X_i \geq 0$. Denote the class of N by $\mathcal{N}$. From the alternating sum of the dimensions of the cohomology groups of $0 \to -\mathcal{N} \to \mathcal{O}_X \to \mathcal{O}_N \to 0$, using Riemann-Roch to collect the terms corresponding to $-\mathcal{N}$, we know that $(N^2 + K_X \cdot N)/2 = h^1(N, \mathcal{O}_N) - h^0(N, \mathcal{O}_N)$ (which, when N is reduced and irreducible, is just adjunction). But $h^1(N, \mathcal{O}_N) = 0$ by hypothesis, and clearly $h^0(N, \mathcal{O}_N) > 0$, forcing the contradiction $N^2 + K_X \cdot N \leq -2$.]

Now let $\mathcal{C} = \mathcal{F} \otimes \mathcal{O}_N$; since $\mathcal{F}$ meets every component of N nonnegatively, $\mathcal{C}$ has nonnegative degree on each component X_i. Let $N' = N - X_0$ and let $\mathcal{C}' = \mathcal{F} \otimes \mathcal{O}_{N'}$ be the restriction of $\mathcal{C}$ to N'. By induction, we may assume $\mathcal{C}'$ is effective (i.e., $h^0(N', \mathcal{F}) > 0$)

and regular (i.e., $h^1(N', \mathcal{F}) = 0$).

Next consider the diagram obtained from

$$\begin{array}{ccccccccc} 0 & \to & -\mathcal{N} & \to & \mathcal{O}_X & \to & \mathcal{O}_N & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & -\mathcal{N}' & \to & \mathcal{O}_X & \to & \mathcal{O}_{N'} & \to & 0 \end{array}$$

by tensoring by $\mathcal{F}$, where the vertical map $\mathcal{O}_N \to \mathcal{O}_{N'}$ is the obvious quotient defining N' as a subscheme of N. Taking cohomology of $0 \to \mathcal{K} \to \mathcal{C} \to \mathcal{C}' \to 0$ (where $\mathcal{K}$ is the kernel of $\mathcal{C} \to \mathcal{C}'$) and using regularity of $\mathcal{C}'$, it is enough to show $h^1(N, \mathcal{K}) = 0$, since then $h^i(N', \mathcal{F}) = h^i(N, \mathcal{F})$ for $i = 1, 2$.

By the snake lemma, $\mathcal{K}$ is isomorphic to the restriction to N of the cokernel of $\mathcal{F} - \mathcal{N} \to \mathcal{F} - \mathcal{N}'$, and this restriction is just extension by 0 to N of $(\mathcal{F} - \mathcal{N}') \otimes \mathcal{O}_{X_0}$. I.e., we only need to show that $h^1(X_0, \mathcal{F} - \mathcal{N}') = 0$, and since X_0 is smooth and rational, this follows if $(\mathcal{F} - \mathcal{N}') \cdot X_0 \geq -1$. But $(\mathcal{F} - \mathcal{N}') \cdot X_0 \geq -\mathcal{N}' \cdot X_0$ and $-\mathcal{N}' \cdot X_0 = -(N - X_0) \cdot X_0$, so, using $N \cdot X_0 \leq 1 + X_0^2$ (established above), $-(N - X_0) \cdot X_0 \geq -1$, as required. $\diamondsuit$

We now prove Theorem I.1:

Proof: Because we assume the points are general, we may assume that no three are collinear, no six lie on a conic, and, if $n = 8$, that there is no cubic passing through all eight with a singularity at one of the points. Now by Theorem 1 of [D], $-K_X$ is ample, hence $K_X^2 > 0$, and, by Lemma II.1(a), $-K_X \in$ EFF.

(a) First we show that EFF is generated by NEF and by the classes of the exceptional curves. Let $\mathcal{C}$ be the class of a reduced, irreducible curve $C \subset X$ with $C^2 < 0$. By adjunction, $C^2 = -K_X \cdot C + 2g - 2$, where g is the genus of C, and, since $-K_X$ is ample, we see that $C^2 = K_X \cdot C = -1$, so C is exceptional. Thus the only curves on X of negative self-intersection are exceptional curves. Hence by ampleness of $-K_X$ and Lemma II.1, any reduced, irreducible curve C which is not exceptional is fixed component free and thus numerically effective. By Corollary II.4, NEF $\subset$ EFF, so it follows that EFF is generated by numerically effective classes and exceptional classes. (The restriction in the statement of Theorem I.1(a) to numerically effective classes with $\mathcal{F}^2 - K_X \cdot \mathcal{F} \geq 0$ is superfluous; it is required to have a single statement which conjecturally remains valid for $n > 9$, since NEF $\subset$ EFF can fail for blowings up of $\mathbf{P}^2$ at 10 or more points.)

We now prove that $\mathcal{F} \in$ NEF if and only if $\mathcal{F}^2 \geq 0$, $\mathcal{F} \cdot \mathcal{E}_0 \geq 0$ and $\mathcal{F} \cdot \mathcal{E} \geq 0$ whenever $\mathcal{E}$ is the class of an exceptional curve. By Proposition II.3, a numerically effective class $\mathcal{F}$ has nonnegative self-intersection, and clearly $\mathcal{F} \cdot \mathcal{E}_0 \geq 0$ and $\mathcal{F} \cdot \mathcal{E} \geq 0$ for every class $\mathcal{E}$ of an exceptional curve. We now must show the converse. If $n = 0$, then EFF consists of nonnegative multiples of the class $\mathcal{E}_0$ of a line, and the result follows. If $n = 1$, then EFF is generated by the class $\mathcal{E}_1$ of the single blow up and by $\mathcal{E}_0 - \mathcal{E}_1$. Thus NEF is generated by $\mathcal{E}_0 - \mathcal{E}_1$ and by $\mathcal{E}_0$, and the result follows directly. If $1 < n \leq 8$, it suffices to show that a positive multiple of any effective class lies in the monoid generated by the exceptional classes.

In fact, if X is any blowing up of $\mathbf{P}^2$ at $n \geq 2$ points such that the only integral curves of negative self-intersection are exceptional curves, then it turns out to be true that NEF lies in the monoid generated by $-K_X$ and the exceptional curves. One can see this, *mutatis mutandis*, using the method of proof of Corollary 2.4 of [H1]; there being no curves of negative self-intersection but exceptional curves can substitute for the general hypothesis of [H1] that there be a reduced and irreducible anticanonical divisor. The result is that if $\mathcal{F}$ is numerically effective, then with respect to some exceptional configuration $\{\mathcal{E}_0, \dots, \mathcal{E}_n\}$, $\mathcal{F}$ is in the monoid generated by $\mathcal{E}_0$, $\mathcal{E}_0 - \mathcal{E}_1$, $2\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2$, and $-K_i$, $i > 2$, where $-K_i = -K_X + \mathcal{E}_{i+1} + \cdots + \mathcal{E}_n$. But for $n \geq 2$, $\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2$ is exceptional and so $\mathcal{E}_0$, $\mathcal{E}_0 - \mathcal{E}_1$ and $2\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2$ are sums of exceptional classes. Also, since $-K_X = 3\mathcal{E}_0 - \mathcal{E}_1 - \cdots - \mathcal{E}_n$, we have $-K_3 = (\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2) + (\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_3) + (\mathcal{E}_0 - \mathcal{E}_2 - \mathcal{E}_3) + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3$, $-K_4 = (\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2) + (\mathcal{E}_0 - \mathcal{E}_3 - \mathcal{E}_4) + (\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2) + \mathcal{E}_1 + \mathcal{E}_2$, $-K_5 = (\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2) + (\mathcal{E}_0 - \mathcal{E}_3 - \mathcal{E}_4) + (\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_5) + \mathcal{E}_1$, $-K_6 = (\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_2) + (\mathcal{E}_0 - \mathcal{E}_3 - \mathcal{E}_4) + (\mathcal{E}_0 - \mathcal{E}_5 - \mathcal{E}_6)$, $-K_7 = (2\mathcal{E}_0 - \mathcal{E}_1 - \cdots - \mathcal{E}_5) + (\mathcal{E}_0 - \mathcal{E}_1 - \mathcal{E}_5) + \mathcal{E}_1$, and $-2K_8 = (3\mathcal{E}_0 - 2\mathcal{E}_1 - \mathcal{E}_2 - \cdots - \mathcal{E}_7) + (3\mathcal{E}_0 - \mathcal{E}_2 - \cdots - \mathcal{E}_7 - 2\mathcal{E}_8)$.

(b) Let $\mathcal{F} \in$ NEF; by Proposition II.3 we know $h^2(X, \mathcal{F}) = 0$ and by Corollary II.4 we know $\mathcal{F} \in$ EFF. Now we must show $h^1(X, \mathcal{F}) = 0$. But since $-K_X$ is ample and effective and $\mathcal{F}$ is effective and numerically effective, we see that $\mathcal{F} - K_X \in$ NEF and that $(\mathcal{F} - K_X)^2 > 0$, so $h^1(X, \mathcal{F}) = h^1(X, -(\mathcal{F} - K_X)) = 0$ by duality and Ramanujam's vanishing theorem (see the first paragraph [R, Theorem, p. 121], which holds in all characteristics). ♢

III. Applications and Remarks

The results of the previous section now allow us to prove our main result, and from it to obtain the fact that any $n \leq 8$ essentially distinct points are in good position (the meaning of which we discuss in remarks below).

Theorem III.1: *Let $\mathcal{F} \in$ NEF be a class on a smooth projective rational surface X with $K_X^2 > 0$. Then $\mathcal{F} \in$ EFF, and $h^1(X, \mathcal{F}) = h^2(X, \mathcal{F}) = 0$.*

Proof: By Proposition II.3, we have $h^2(X, \mathcal{F}) = 0$. By Lemma II.1, we have $-K_X \in$ EFF (and indeed $h^0(X, -K_X) > 1$), so by Corollary II.4, we have $\mathcal{F} \in$ EFF. We must show $h^1(X, \mathcal{F}) = 0$. This is clear (since rational surfaces are regular) if $\mathcal{F}$ is trivial, so assume $\mathcal{F}$ is not trivial. Write $\mathcal{F} = \mathcal{L} + \mathcal{N}$, where $\mathcal{N}$ is the fixed part of $\mathcal{F}$ and $\mathcal{L}$ is its fixed component free part. By Lemma II.5, $h^1(X, \mathcal{L}) = 0$.

Now let N be a section of $\mathcal{N}$. Since $-K_X$ is not fixed but $\mathcal{N}$ is, $h^0(X, \mathcal{N} + K_X) = 0$ is clear. By duality, $h^2(X, -\mathcal{N}) = 0$, and so taking cohomology of $0 \to -\mathcal{N} \to \mathcal{O}_X \to \mathcal{O}_N \to 0$ we see that $h^1(N, \mathcal{O}_N) = 0$. Thus $h^1(N, \mathcal{F}) = 0$ by Lemma II.6, while $h^1(X, \mathcal{L}) = 0$ by Lemma II.5, so tensoring the foregoing short exact sequence by $\mathcal{F}$ we obtain $0 \to \mathcal{L} \to \mathcal{F} \to \mathcal{O}_N \otimes \mathcal{F} \to 0$, and taking cohomology we conclude $h^1(X, \mathcal{F}) = 0$. ♢

We end with a few remarks, beginning with some background from [H5]. Consider morphisms $X = X_n \to \cdots \to X_0 = \mathbf{P}^2$ where $\pi_i : X_i \to X_{i-1}$ is the blowing up of a point $p_i \in X_{i-1}$. We call such an ordered set $p_1, \dots, p_n$ of points n *essentially distinct* points of $\mathbf{P}^2$. If none of the points is infinitely near another, we have the usual notion of distinct points in $\mathbf{P}^2$. We say that $p_1, \dots, p_n$ are in *good position* if the resulting surface X has no irregular, effective, numerically effective divisor classes. As discussed in the introduction, it is conjectured that n "sufficiently general" essentially distinct points are in good position.

In order to better understand this problem, we undertook in [H5] to determine the locus, in the space of all sets of n essentially distinct points, of those sets which are in good position, beginning with small n. Although for $n < 10$ it was known that n sufficiently general points are in good position (Theorem I.1 being a proof for $n < 9$), the question of precisely which sets are in good position had not been considered. Also note that it is sensible to speak of the space of all n essentially distinct points of $\mathbf{P}^2$, since we have a fine moduli space $\mathcal{B}_n$ and a universal family $\mathcal{F}_n \to \mathcal{B}_n$, constructed in [H4] for each n, of blowings up of $\mathbf{P}^2$ at sets of n essentially distinct points.

By a completely different approach than is used here, [H5] showed that any 6 or fewer essentially distinct points are in good position and that any 8 or fewer distinct points are in good position, and raised the conjecture that any 8 or fewer essentially distinct points are in good position. Since any blowing up X of $\mathbf{P}^2$ at $n \leq 8$ essentially distinct points has $K_X^2 > 0$, Theorem III.1 settles this conjecture affirmatively, giving:

Corollary III.2: *Any $n \leq 8$ essentially distinct points of $\mathbf{P}^2$ are in good position.*

While the question of which sets of $n \geq 10$ essentially distinct points of $\mathbf{P}^2$ are in good position remains mysterious, the following result (based on a more extensive examination in [H6], extending our results here, of rational surfaces with an effective anticanonical divisor) implies that 9 essentially distinct points are in good position if and only if, on the surface X obtained by blowing up the points, either $-K_X$ is not numerically effective, or $-K_X \in$ NEF but no positive multiple of $-K_X$ moves in a positive dimensional linear system (by Lemma II.1, the latter is equivalent to $h^1(X, -rK_X) = 0$ for all $r > 0$).

Corollary III.3: *Let X be a smooth projective rational surface with $K_X^2 = 0$. Then X supports an irregular effective numerically effective class on X if and only if $-rK_X$ is numerically effective and irregular for some $r > 0$.*

Proof: Since X is rational, $0 = h^0(X, 2K_X) = h^2(X, -K_X)$, so by Lemma II.1(a), $-K_X \in$ EFF, hence numerically effective classes are effective by Corollary II.4. By Corollary III.3(b) [H6], a numerically effective class $\mathcal{F}$ is regular if $-K_X \cdot \mathcal{F} > 0$. Thus X supports an irregular numerically effective class if and only if there is an irregular numerically effective class in the subgroup $K_X^\perp \subset \mathrm{Pic}(X)$ of classes perpendicular to K_X. But by Lemma II.4 [H6], any numerically effective class in $K_X^\perp$ is a nonnegative multiple of $-K_X$. Since $\mathcal{O}_X$ is always regular, X has a numerically effective but irregular class if and only if $-rK_X$ is numerically effective and irregular for some $r > 0$. $\diamondsuit$

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