Linear systems with multiple base points in $\mathbf{P}^2$

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Abstract: Conjectures for the Hilbert function $h(n;m)$ and minimal free resolution of the mth symbolic power $I(n;m)$ of the ideal of n general points of $\mathbf{P}^2$ are verified for a broad range of values of m and n where both m and n can be large, including (in the case of the Hilbert function) for infinitely many m for each square $n > 9$ and (in the case of resolutions) for infinitely many m for each even square $n > 9$. All previous results require either that n be small or be a square of a special form, or that m be small compared to n. Our results are based on a new approach for bounding the least degree among curves passing through n general points of $\mathbf{P}^2$ with given minimum multiplicities at each point and for bounding the regularity of the linear system of all such curves. For simplicity, we work over the complex numbers.

I. Introduction

Consider the ideal $I(n;m) \subset R = \mathbf{C}[\mathbf{P}^2]$ generated by all forms having multiplicity at least m at n given general points of $\mathbf{P}^2$. This is a graded ideal, and thus we can consider the Hilbert function $h(n;m)$ whose value at each nonnegative integer t is the dimension $h(n;m)(t) = \dim I(n;m)_t$ of the homogeneous component $I(n;m)_t$ of $I(n;m)$ of degree t. It is well known that $h(n;m)(t) \geq \max(0, \binom{t+2}{2} - n\binom{m+1}{2})$, with equality for t sufficiently large. Denote by $\alpha(n;m)$ the least degree t such that $h(n;m)(t) > 0$ and by $\tau(n;m)$ the least degree t such that $h(n;m)(t) = \binom{t+2}{2} - n\binom{m+1}{2}$; we refer to $\tau(n;m)$ as the *regularity* of $I(n;m)$.

For $n \leq 9$, the Hilbert function [N2] and minimal free resolution [H2] of $I(n;m)$ are known. For $n > 9$, there are in general only conjectures:

Conjecture I.1: *Let $n \geq 10$ and $m \geq 0$; then:*

(a) $\alpha(n;m) \geq m\sqrt{n}$;

(b) $h(n;m)(t) = \max(0, \binom{t+2}{2} - n\binom{m+1}{2})$ *for each integer $t \geq 0$; and*

(c) *the minimal free resolution of $I(n;m)$ is an exact sequence*

$$0 \to R[-\alpha-2]^d \oplus R[-\alpha-1]^c \to R[-\alpha-1]^b \oplus R[-\alpha]^a \to I(n;m) \to 0,$$

where $\alpha = \alpha(n;m)$, $a = h(n;m)(\alpha)$, $b = max(h(n;m)(\alpha+1) - 3h(n;m)(\alpha), 0)$, $c = max(-h(n;m)(\alpha+1) + 3h(n;m)(\alpha), 0)$, $d = a+b-c-1$, and $R[i]^j$ is the direct sum of j copies of the ring $R = \mathbf{C}[\mathbf{P}^2]$, regarded as an R-module with the grading $R[i]_k = R_{k+i}$.

Note that Conjecture I.1(c) implies Conjecture I.1(b) which implies Conjecture I.1(a). Conjecture I.1(a) was posed in [N1] in the form "$\alpha(n;m) > m\sqrt{n}$ for $n \geq 10$ and $m > 0$", together with a proof in case $n > 9$ is a square. Conjecture I.1(c) was posed in [H2], together with a determination of the resolution for $n \leq 9$. Conjecture I.1(b) is a special case of more general conjectures posed in different but equivalent forms by a number of people. In particular, given general points $p_1, \ldots, p_n \in \mathbf{P}^2$, let $\mathbf{m} = (m_1, \ldots, m_n)$ be any sequence of nonnegative integers, and define $I(\mathbf{m})$ to be the ideal generated by all forms having multiplicity at least m_i at p_i. We can in the obvious and analogous way define $\alpha(\mathbf{m})$, $\tau(\mathbf{m})$ and $h(\mathbf{m})$. Whereas in this generality no conjecture for the minimal free resolution of $I(\mathbf{m})$ has yet been posed (but see [FHH] for a solution when $n \leq 8$), equivalent conjectures for $h(\mathbf{m})$ have been posed in [H1], [Gi] and [Hi1]. (Ciliberto and Miranda have recently pointed out that these conjectures are also equivalent to what seemed to be a weaker conjecture posed in [S]; see [H4] for a discussion.)

These conjectures for $h(\mathbf{m})$ have been verified in certain special cases: for $n \leq 9$ by [N2]; for any n as long as $m_i \leq 4$ for all i by [Mg]; and by [AH] for any m_i as long as the maximum of the m_i is sufficiently small compared to the number of points for which $m_i > 0$. In addition, Conjecture I.1(b) has been verified by [CM2] as long as $m \leq 12$. The only result up to now with $n \geq 10$ and arbitrarily large multiplicities had been that of [E2], which verifies Conjecture I.1(b) for all m as long as n is a power of 4.

Similarly, [E1] verifies Conjecture I.1(a) as long as m is no more than about $\sqrt{n}/2$, and [I] verifies Conjecture I.1(c) for $m = 2$ (the case of $m = 1$ having been done by [GGR]), while [HHF] applies the result of [E2] to verify Conjecture I.1(c) when n is a power of 4, as long as m is not too small.

In this paper we obtain substantial improvements on these prior results for all three parts of Conjecture I.1. For example, we have:

Corollary I.2: *Let $n \geq 10$. Then:*
(a) *Conjecture I.1(a) holds as long as $m \leq (n - 5\sqrt{n})/2$;*
(b) *Conjecture I.1(b) holds for infinitely many m for each square $n \geq 10$; and*
(c) *Conjecture I.1(c) holds for infinitely many m for each even square $n \geq 10$.*

We also verify Conjecture I.1(a,b,c) for many other values of m and n (see Corollary IV.1, Corollary V.1 and Corollary VI.1). Although the nature of our approach makes it difficult to give a simple description of all m and n which we can handle, see Figures 1–5 for graphical representations of some of our results.

The key to our results is our development of new bounds for $\alpha(\mathbf{m})$ and $\tau(\mathbf{m})$: Sufficiently good bounds determine these quantities exactly, which in many situations is sufficient to also determine $h(\mathbf{m})$ and the minimal free resolution of $I(\mathbf{m})$. Our bounds are algorithmic; given $\mathbf{m}$ and any positive integers d and $r \leq n$, we give an algorithm for computing bounds on $\alpha(\mathbf{m})$ and $\tau(\mathbf{m})$. An analysis of our algorithm leads to explicit formulas for these bounds in certain cases.

To write down these formulas, given $d > 0$, $0 < r \leq n$ and $\mathbf{m} = (m_1, \ldots, m_n)$, define integers u and ρ via $M_n = ur + \rho$, where $u \geq 0$, $0 < \rho \leq r$ and $M_i = m_1 + \cdots + m_i$ for each $1 \leq i \leq n$. Note that we can equivalently define $u = \lceil M_n/r \rceil - 1$ and $\rho = M_n - ru$. Also, given an integer $0 < r \leq n$, we say that $(m_1, \ldots, m_n)$ is r-*semiuniform* if $m_r + 1 \geq m_1 \geq m_2 \geq \cdots \geq m_n \geq 0$. Note that a nonincreasing sequence $(m_1, \ldots, m_n)$ of nonnegative integers is r-semiuniform if and only if m_i is either m_r or $m_r + 1$ for every $1 \leq i \leq r$; thus, for example, $(6, 6, 6, 5, 5, 5, 5, 4, 4, 4)$ is r-semiuniform for each r up to 7, but not for $r = 8, 9$ or 10. We now have:

Theorem I.3: *Given integers $0 < d$ and $0 < r \leq n$, let $\mathbf{m} = (m_1, \ldots, m_n)$ be r-semiuniform, define M_i, u and ρ as above, denote the genus $(d-1)(d-2)/2$ of a plane curve of degree d by g and let s be the largest integer such that we have both $(s+1)(s+2) \leq 2\rho$ and $0 \leq s < d$.*

(a) *If $r \leq d^2$, then*
  (i) *$\alpha(\mathbf{m}) \geq 1 + min(\lfloor (M_r + g - 1)/d \rfloor, s + ud)$ whenever $d(d+1)/2 \leq r$, while*
  (ii) *$\tau(\mathbf{m}) \leq max(\lceil (\rho + g - 1)/d \rceil + ud, ud + d - 2)$.*

(b) *If $2r \geq n + d^2$, then*
  (i) *$\alpha(\mathbf{m}) \geq s + ud + 1$, and*
  (ii) *$\tau(\mathbf{m}) \leq max(\lceil (M_r + g - 1)/d \rceil, ud + d - 2)$.*

(c) *Say for some m we have $m_i = m$ for all i, and that $rd(d+1)/2 \leq r^2 \leq d^2 n$; then $\alpha(n; m) \geq 1 + \min(\lfloor (mr + g - 1)/d \rfloor, s + ud)$.*

In Section II, we develop the results needed to state and analyze our algorithm. In Section III we prove Theorem I.3. In Section IV we apply Theorem I.3 to prove results less easily stated but stronger than Corollary I.2(a), from which Corollary I.2(a) is an easy consequence. Similarly, Corollary I.2(b) is an immediate consequence of more comprehensive results that we deduce from Theorem I.3 in Section V, and Corollary I.2(c) is an immediate consequence of more comprehensive results that we deduce in Section VI from Theorem I.3 using [HHF].

Since it is hard to easily describe all of the cases that our method handles, we include for this purpose some graphs in Section VII, together with some explicit comparisons of our bounds on α and τ with previously known bounds.

II. Algorithms

In this section we derive algorithms giving bounds on $\alpha(\mathbf{m})$ and $\tau(\mathbf{m})$. It is of most interest to give lower bounds for α and upper bounds for τ, since upper bounds for α and lower bounds for τ are known which are conjectured to be sharp. (See [H4] for a discussion.)

Our approach, which combines those of [H3, HHF, R1, R2], involves a specialization of the n points as in [H3] (which in turn was originally inspired by that of [R1]), together with properties of linear series on curves. Recall that a flex for a linear series V of dimension a on a curve C is a point $p \in C$ such that $V - (a+1)p$ is not empty. In Lemma II.1 we use the known result (see p. 235, [Mr]) that the set of flexes of a linear series is finite; it is the only place that we need the characteristic to be zero (although, of course, everything else refers to this Lemma). In positive characteristics, complete linear series can indeed have infinitely many flexes [Ho].

Before deriving our algorithms, we need two lemmas.

Lemma II.1: *Let C be an irreducible plane curve of degree d, so $g = (d-1)(d-2)/2$ is the genus of C, and let q be a general point of C. Take $D = tL_C - vq$, where $t \geq 0$ and $v \geq 0$ are integers and L_C is the restriction to C of a line L in $\mathbf{P}^2$.*

(1) If $t \geq d-2$ then $h^0(C, D) = 0$ for $t \leq (v+g-1)/d$ and $h^1(C, D) = 0$ for $t \geq (v+g-1)/d$.

(2) If $t < d$ then $h^0(C, D) = 0$ for $(t+1)(t+2) \leq 2v$.

Proof: We have $L, C \subset \mathbf{P}^2$. The linear system $|tL_C|$ is the image of $|tL|$ under restriction to C. Note that $h^1(C, tL_C) = 0$ if (and only if) $t \geq d-2$, in which case $h^0(C, tL_C) = td-g+1$, whereas, for $t < d$, $h^0(C, tL_C) = h^0(\mathbf{P}^2, tL) = (t+1)(t+2)/2$.

Since $h^1(C, tL_C) = 0$ for $t \geq d-2$, as long as vq imposes independent conditions on $|tL_C|$ (i.e., $h^0(C, tL_C - vq) = h^0(C, tL_C) - v$), then $h^1(C, tL_C - vq) = 0$ too. Also, if we show that $h^1(C, tL_C - vq) = 0$, then it is easy to see that $h^1(C, tL_C - v'q) = 0$ for all $v' \leq v$, and if we show that $h^0(C, tL_C - vq) = 0$, then it is easy to see that $h^0(C, tL_C - v'q) = 0$ for all $v' \geq v$. So it is enough to show $h^0(C, tL_C - vq) = 0$ for $v = h^0(C, tL_C)$. Since q is general in C, we can assume it is not a flex of $|tL_C|$, and therefore the claim follows. $\diamondsuit$

Consider n distinct points $p_1, \ldots, p_n$ of $\mathbf{P}^2$ and let X be the blow up of the points. More generally, we can allow the possibility that some of the points are infinitely near by taking $p_1 \in \mathbf{P}^2 = X_0$, $p_2 \in X_1$, ..., $p_n \in X_{n-1}$, where X_i, for $0 < i \leq n$, is the blow up of X_{i-1} at p_i, and we take $X = X_n$. Given integers t and $m_1 \geq m_2 \geq \cdots \geq m_n \geq 0$, we denote by F_t the divisor $F_t = tL - m_1E_1 - \cdots - m_nE_n$ on X, where E_i is the divisorial inverse image of p_i under the blow up morphisms $X = X_n \to X_{n-1} \to \cdots \to X_{i-1}$, and L is the pullback to X of a general line in $\mathbf{P}^2$. Note that the divisor classes $[L], [E_1], \ldots, [E_n]$ give a basis for the divisor class group $\mathrm{Cl}(X)$ of X.

Now, given positive integers $r \leq n$ and d, we choose our points $p_1, p_2, \ldots, p_n$ such that p_1 is a general smooth point of an irreducible plane curve C' of degree d, and then choose points $p_2, \ldots, p_n$ so that p_i is infinitely near p_{i-1} for $i \leq n$ and so that p_i is a point of the proper transform of C' on X_{i-1} for $i \leq r$ (more precisely, so that $[E_i - E_{i+1}]$ is the class of an effective, reduced and irreducible divisor for $0 < i < n$ and so that the class of the proper transform of C' to X is $[dL - E_1 - \cdots - E_r]$). Let C denote the proper transform of C' in X.

Define divisors D_j and D'_j such that $D_0 = F_t$, $D'_j = D_j - (dL - E_1 - \cdots - E_r)$, and such that D_{j+1} is obtained from D'_j by *unloading* multiplicities (i.e., if $D'_j = a_0L - a_1E_1 - \cdots - a_nE_n$, then we permute the $a_1, a_2, \ldots, a_n$ so that $a_1 \geq a_2 \geq \cdots \geq a_n$ and set to 0 each which is negative).

Part (1) of the following lemma is used in our algorithms. Parts (2) and (3) are used in the proof of Theorem I.3(a,b).

Lemma II.2: *Let $r \leq n$ and d be positive integers. Given $F_t = tL - m_1E_1 - \cdots - m_nE_n$*

with $m_1 \geq m_2 \geq \cdots \geq m_n \geq 0$*, define* C*,* D_j *and* D'_j *for* $j \geq 0$ *as above.*

(1) For every $j > 0$*,* $h^i(X, D'_{j-1}) = h^i(X, D_j)$*,* $i = 0, 1$*.*

(2) If $r \leq d^2$*, then* $D_j \cdot C \leq D_{j-1} \cdot C$ *for every* $j > 0$*.*

(3) If $2r \geq n + d^2$*, then* $D_j \cdot C \geq D_{j-1} \cdot C$ *as long as* $D_{j-1} \cdot E_r > 0$*.*

Proof: Write $D'_{j-1} = a_0 L - a_1 E_1 - \cdots - a_n E_n$. Because of the definition of F_t and D_i, we have either $a_1 \geq a_2 \geq \cdots \geq a_n \geq -1$, or $a_1 \geq a_2 \geq \cdots \geq a_i = a_{i+1} - 1$ and $a_{i+1} \geq \cdots \geq a_n \geq -1$ for some $i \leq r$. Therefore, the unloading procedure leading from D'_{j-1} to D_j consists in a number of unloading steps, each of which either transposes a_k and a_{k+1} (whenever $a_k = a_{k+1} - 1$) or sets a_n to 0 (whenever in the course of the transpositions we find $a_n = -1$). If we denote the proper transform of the exceptional divisor of blowing up p_k by $\tilde{E}_k$ (hence $[\tilde{E}_k] = [E_k - E_{k+1}]$ for $k < n$ and $\tilde{E}_n = E_n$), this is the same as iteratively subtracting $\tilde{E}_k$ from D'_{j-1} whenever $D'_{j-1} \cdot \tilde{E}_k = -1$, and it is enough to show that doing so does not affect cohomology in order to prove (1).

Consider the exact sequence $0 \to \mathcal{O}_X(D'_{j-1} - \tilde{E}_k) \to \mathcal{O}_X(D'_{j-1}) \to \mathcal{O}_X(D'_{j-1}) \otimes \mathcal{O}_{\tilde{E}_k} \to 0$. Since $\tilde{E}_k \cong \mathbf{P}^1$ and $D'_{j-1} \cdot \tilde{E}_k = -1$, we have $h^i(\mathcal{O}_X(D'_{j-1}) \otimes \mathcal{O}_{\tilde{E}_k}) = 0$. Thus, taking cohomology of the sequence, we see that the cohomology of $\mathcal{O}_X(D'_{j-1} - \tilde{E}_k)$ and $\mathcal{O}_X(D'_{j-1})$ coincide, as claimed.

To prove (2), let $D'_{j-1} = a_0 L - a_1 E_1 - \cdots - a_n E_n$ as above and observe that $D'_{j-1} \cdot C = D_{j-1} \cdot C - d^2 + r$. Passing from D'_{j-1} to D_j by unloading might increase some of the coefficients a_i with $0 < i \leq r$ but cannot decrease any of these coefficients and hence cannot increase the intersection with C; i.e., $D_j \cdot C \leq D'_{j-1} \cdot C = D_{j-1} \cdot C - d^2 + r \leq D_{j-1} \cdot C$.

To prove (3), let $D'_{j-1} = a_0 L - a_1 E_1 - \cdots - a_n E_n$ as above. If $D_{j-1} \cdot E_r > 0$, then $a_i \geq 0$ for all $i > 0$ and $a_i + 1 \geq a_j$ for all $0 < i \leq r < j \leq n$. Thus passing from D'_{j-1} to D_j may involve swapping some of the coefficients a_i, $0 < i \leq r$, with some of the coefficients a_j, $j > r$, but there are only $n - r$ coefficients a_j with $j > r$, each of which is at most 1 bigger than the least coefficient a_i with $0 < i \leq r$, so passing from D'_{j-1} to D_j can decrease the intersection with C by at most $n - r$; i.e., $D_j \cdot C \geq D'_{j-1} \cdot C - (n - r) = D_{j-1} \cdot C - d^2 + r - (n - r) \geq D_{j-1} \cdot C$. ♢

For Theorem I.3(c) we need a slightly more general version of the preceding lemma, restricted however to the case of uniform multiplicities:

Lemma II.3: *Let* $r \leq n$ *and* d *be positive integers. Let* $F_t = tL - mE_1 - \cdots - mE_n$*, with respect to which we take* C*,* D_i *and* D'_i *for all* $i \geq 0$ *as in Lemma II.2, and let* ω' *be the least* i *such that* $D_i \cdot E_i = 0$ *for all* $i > 0$*. Then for all* $0 \leq i \leq \omega' - 1$ *we have*

$$i\left(\frac{r^2}{n} - d^2\right) - \left(r - \frac{r^2}{n}\right) \leq D_i \cdot C - D_0 \cdot C \leq i\left(\frac{r^2}{n} - d^2\right).$$

Proof: Let $A_0 = 0$, and for $0 < k \leq n$ let $A_k = -E_1 - \cdots - E_k$ and define t_i to be $D_i \cdot L$. For $0 \leq i < \omega'$, it is not hard to check that $D_i = (t_0 - id)L - (m - i + q)E_1 - \cdots - (m - i + q)E_n + A_\rho$, where $i(n - r) = qn + \rho$ with $0 \leq \rho < n$, and therefore

$$D_i \cdot C - D_0 \cdot C = i(r - d^2) - rq + A_\rho \cdot C.$$

On the other hand, $A_\rho \cdot C = -\min(\rho, r)$, and it is easy to see that

$$i(n-r)\frac{r}{n} \leq rq + \min(\rho, r) \leq i(n-r)\frac{r}{n} + r - \frac{r^2}{n},$$

from which the claim follows. $\diamondsuit$

We now derive our algorithms. Let X be obtained by blowing up n general points of $\mathbf{P}^2$, let $F_t = tL - m_1E_1 - \cdots - m_nE_n$, where we assume that $m_1 \geq \cdots \geq m_n \geq 0$, and choose any integers d and r such that $d > 0$ and $0 < r \leq n$. Next, specialize the points as in Lemma II.2. We then have the specialized surface X'. It is convenient to denote the basis of the divisor class group of X' corresponding to the specialized points also by L, $E_1, \ldots, E_n$, since it will always be clear whether we are working on X' or on X. By semicontinuity, we know $h^i(X', F_t) \geq h^i(X, F_t)$, so for any t with $h^i(X', F_t) = 0$ we also have $h^i(X, F_t) = 0$.

As in Lemma II.2, we have on X' the curve C with class $[dL - E_1 - \cdots - E_r]$ and genus $g = (d-1)(d-2)/2$ and we have the sequence of divisors D_j, where $D_0 = F_t$, $D'_j = D_j - (dL - E_1 - \cdots - E_r)$, and D_{j+1} is obtained from D'_j as above.

Whenever $h^0(X', F_t) = 0$, we get the bound $t + 1 \leq \alpha(m_1, \ldots, m_n)$, and whenever $h^1(X', F_t) = 0$, we get the bound $t \geq \tau(m_1, \ldots, m_n)$. But, using the exact sequences $0 \to \mathcal{O}_{X'}(D'_j) \to \mathcal{O}_{X'}(D_j) \to \mathcal{O}_{X'}(D_j) \otimes \mathcal{O}_C \to 0$ and Lemma II.2(1), we see that $h^0(X', F_t) = 0$ if for some $i = I$ we have $h^0(X', D_I) = 0$ and for all $0 \leq i < I$ we have $h^0(C, \mathcal{O}_{X'}(D_i) \otimes \mathcal{O}_C) = 0$. Similarly, $h^1(X', F_t) = 0$ if for some $j = J$ we have $h^1(X', D_J) = 0$ and for all $0 \leq j < J$ we have $h^1(C, \mathcal{O}_{X'}(D_j) \otimes \mathcal{O}_C) = 0$. But, $\mathcal{O}_{X'}(D_j) \otimes \mathcal{O}_C = \mathcal{O}_{X'}((t - jd)L) \otimes \mathcal{O}_C(-v_jq)$, where $v_j = (t - jd)d - D_j \cdot C$ and q is the point of C infinitely near to p_1. Thus we can apply the criteria of Lemma II.1 to control $h^0(C, \mathcal{O}_{X'}(D_i) \otimes \mathcal{O}_C)$ for $0 \leq i < I$ and $h^1(C, \mathcal{O}_{X'}(D_j) \otimes \mathcal{O}_C)$ for $0 \leq j < J$.

In particular, for a given value of t we have $h^0(X', D_I) = 0$ if we take I to be the least i such that $D_i \cdot (L - E_1) < 0$. Then, if t is not too large, for all $0 \leq i < I$ we also have either $D_i \cdot C \leq g - 1$ and $t - id \geq d - 2$, or $t - id < 0$, or $t - id < d$ and $(t - id + 1)(t - id + 2) \leq 2v_i$, which guarantees by Lemma II.1 that $h^0(C, D_i) = 0$ for all $0 \leq i < I$. The largest such t then gives a lower bound for $-1 + \alpha(m_1, \ldots, m_n)$; i.e., $t + 1 \leq \alpha(m_1, \ldots, m_n)$.

Similarly, let J be the least j such that D_j is a multiple of L (i.e., such that if $D_j = a_0L - a_1E_1 - \cdots - a_nE_n$, then $a_1 = \cdots = a_n = 0$, in which case we have $h^1(X', D_j) = 0$). If we have chosen t sufficiently large, then $D_j \cdot L \geq d - 2$ and $D_j \cdot C \geq g - 1$, for all $0 \leq j < J$; the least such t then gives an upper bound for $\tau(m_1, \ldots, m_n)$.

III. Proof of Theorem I.3

In this section we prove Theorem I.3, beginning with (a)(i), so assume $d(d+1)/2 \leq r \leq d^2$ and $\mathbf{m} = (m_1, \ldots, m_n)$. To show $1 + t \leq \alpha(\mathbf{m})$ for $t = \min(\lfloor (M_r + g - 1)/d \rfloor, s + ud)$, it is by our algorithm in Section II enough (taking $I = u + 1$) to show that $D_I \cdot (L - E_1) < 0$, and for all $0 \leq i < I$ that either $D_i \cdot C \leq g - 1$ and $t - id \geq d - 2$, or $t - id < 0$, or $0 \leq t - id < d$ and $(t - id + 1)(t - id + 2) \leq 2v_i$, where $v_j = (t - jd)d - D_j \cdot C$, as before.

First, by Lemma II.2(2), $D_0 \cdot C \geq D_1 \cdot C \geq \cdots$, and by hypothesis $t \leq \lfloor (M_r + g - 1)/d \rfloor$, so $g - 1 \geq td - M_r = D_0 \cdot C \geq D_1 \cdot C \geq \cdots$, as required. Also by hypothesis, we have

$t \le s + ud$. It follows that $D_I \cdot L = t - Id$ and hence that $D_I \cdot (L - E_1) < 0$, as required. Thus it is now enough to check that $(t - id + 1)(t - id + 2) \le 2v_i$ for the largest i (call it i'') such that $t - id \ge 0$. If $i'' = I - 1$ we have $t - i''d = t - ud \le s$ by hypothesis and hence $(t - i''d + 1)(t - i''d + 2) \le (s + 1)(s + 2) \le 2\rho = 2v_{I-1}$ by definition of s. If $i'' < I - 1$, we at least have $t - i''d \le d - 1$, so $(t - i''d + 1)(t - i''d + 2) \le d(d + 1)$. But $v_{I-2} \ge r$ by r-semiuniformity, so $M_r = v_0 \ge v_1 \ge \cdots \ge v_{I-2} \ge r \ge v_{I-1} = \rho > v_I = 0$, hence $2v_{i''} \ge 2r \ge d(d + 1) \ge (t - i''d + 1)(t - i''d + 2)$, as we wanted.

We now prove (a)(ii). As always we have $r \le n$; in addition we assume $r \le d^2$. It follows from semiuniformity that D_J is a multiple of L for $J = u+1$, so it suffices to show for $t = \max(\lceil(\rho + g - 1)/d\rceil + ud, ud + d - 2)$ that $D_j \cdot L \ge d - 2$ and $D_j \cdot C \ge g - 1$, for all $0 \le j < J$; it then follows by our algorithm that $t \ge \tau(\mathbf{m})$. But $t \ge ud + d - 2$ ensures that $D_j \cdot L \ge d - 2$ for all $0 \le j < J$, and, since $D_0 \cdot C \ge D_1 \cdot C \ge \cdots \ge D_u \cdot C = (t - ud)d - \rho$ by Lemma II.2, $t \ge \lceil(\rho + g - 1)/d\rceil + ud$ ensures that $D_j \cdot C \ge g - 1$, for all $0 \le j < J$.

Next, consider (b). Now we assume that $2r \ge n + d^2$ and that $\mathbf{m}$ is r-semiuniform. By semiuniformity we have $D_i \cdot E_r > 0$ for $i < u$. Now by Lemma II.2(3) we have $D_0 \cdot C \le D_1 \cdot C \le \cdots \le D_u \cdot C$.

Starting with (b)(i), let $I = u+1$ and $t = s+ud$. It suffices to check that $D_I \cdot (L - E_1) < 0$, and for all $0 \le i < I$ that either $D_i \cdot C \le g - 1$ and $t - id \ge d - 2$, or $t - id < d$ and $(t - id + 1)(t - id + 2) \le 2v_i$. But $D_I \cdot (L - E_1) = t - (u + 1)d = s - d < 0$, as required, so now consider D_u. Here we have $D_u = sL - (E_1 + \cdots + E_\rho)$. But by hypothesis $s < d$ and $(s + 1)(s + 2) \le 2\rho = 2v_u$, as required. Finally, consider D_i for $0 \le i < u$. Then $D_i \cdot L = t - id = s + (u - i)d \ge d$, and $sd - \rho = D_u \cdot C \ge D_i \cdot C$, so if we prove that $s \le (\rho + g - 1)/d$ we will have $D_i \cdot C \le g - 1$ as we want. But it is easy to see that $sd - g + 1 = (s + 1)(s + 2)/2 - (s - d + 1)(s - d + 2)/2 \le (s + 1)(s + 2)/2$, and therefore the hypothesis $2\rho \ge (s + 1)(s + 2)$ implies $\rho \ge sd - g + 1$, so we are done.

Next, we prove (b)(ii). Let $J = u + 1$; then $D_J = (t - Jd)L$ is a multiple of L, as by our algorithm we would want. Now assume in addition that $t \ge \max(\lceil(M_r + g - 1)/d\rceil, ud + d - 2)$. We want to verify that $D_j \cdot L \ge d - 2$ and $D_j \cdot C \ge g - 1$, for all $0 \le j < J$. First consider $j = 0$; since $t \ge (M_r + g - 1)/d$ and $t \ge ud + d - 2 \ge d - 2$, we have $dt - M_r = D_0 \cdot C \ge g - 1$ and $t = D_0 \cdot L \ge d - 2$. As for $0 < j < J$, we have $t - jd \ge (ud + d - 2) - (ud) = d - 2$ and $D_j \cdot C \ge D_0 \cdot C \ge g - 1$, which ends the proof of Theorem I.3(b).

Finally, we prove (c). In the notation of Lemma II.3 and its proof, it is easy to check that $\omega' = \lceil mn/r\rceil = u + 1$, so if $t \le s + ud$, it follows that $t_{\omega'} \le s - d < 0$, and thus $\omega' \ge \omega$, where ω is the least i such that $t_i < 0$. Since $r^2/n - d^2 \le 0$, it follows from Lemma II.3 that $D_i \cdot C \le D_0 \cdot C$ for all $0 \le i \le \omega - 2$. If $t \le \lfloor(mr + g - 1)/d\rfloor$, then $D_i \cdot C \le D_0 \cdot C = td - mr \le g - 1$. To conclude that $\alpha(n; m) \ge t + 1$, it is now enough to check that $(t - i'd + 1)(t - i'd + 2) \le 2v_{i'}$ for $i' = \omega - 1$. If $i' = u$ (i.e., $\omega' = \omega$) we have $t - i'd = t - ud \le s$ by hypothesis and hence $(t - i'd + 1)(t - i'd + 2) \le (s+1)(s+2) \le 2\rho = 2v_{i'}$ by definition of s. If $i' < u$ (so $\omega' > \omega$), by definition of i' we at least have $t - i'd \le d - 1$, so $(t - i'd + 1)(t - i'd + 2) \le d(d + 1)$. But $\omega' > \omega$ implies $v_{i'} \ge r$, and by hypothesis $rd(d+1)/2 \le r^2$ (so $d(d+1) \le 2r$); therefore $2v_{i'} \ge 2r \ge d(d+1) \ge (t - i'd + 1)(t - i'd + 2)$ as we wanted. $\diamondsuit$

IV. Nagata's Conjecture

In this section we prove Corollary I.2(a) as an immediate easy-to-state consequence of our following more involved result. Because Conjecture I.1(a) is known when n is a square, we need not consider that case.

Corollary IV.1: *Given an integer $n > 9$, let $d = \lfloor\sqrt{n}\rfloor$ (hence $d \geq 3$) and $\Delta = n - d^2$. Then $\alpha(n; m) \geq m\sqrt{n}$ holds whenever:*

(a) Δ is odd and $m \leq \max(d(d-3), d(d-2)/\Delta)$, or

(b) $\Delta > 0$ is even, and $m \leq \max(d(d-3)/2, 2d^2/\Delta)$.

Proof: Consider part (a). We first prove $\alpha(n; m) \geq m\sqrt{n}$ if Δ is odd and $m \leq d(d-2)/\Delta$. Apply Theorem I.3(a) with $r = d^2$, $u = m$ and $\rho = m\Delta$; it has to be checked that $md + s + 1 \geq m\sqrt{n}$ and $\lfloor(mr + g - 1)/d\rfloor + 1 \geq m\sqrt{n}$. The first inequality is equivalent to $(s+1)^2 + 2(s+1)md \geq m^2\Delta$. If $s = 0$ then $m\Delta < 3$ and the inequality follows from $d \geq 3$, whereas if $s = d-1$ then $d^2 > d(d-2) \geq m\Delta$ says $d^2 + 2md^2 > m^2\Delta$. In all intermediate cases one has $2d(s+1) \geq (s+2)(s+3)$ and $(s+2)(s+3) > 2\rho = 2m\Delta$ which also imply $(s+1)^2 + 2(s+1)md \geq m^2\Delta$ easily. To prove the second inequality it is enough to see that $md + d/2 - 1 \geq m\sqrt{n}$, which is equivalent to $(d/2-1)^2 + md(d-2) \geq m^2\Delta$, and this follows from $m \leq d(d-2)/\Delta$.

We now prove $\alpha(n; m) \geq m\sqrt{n}$ if Δ is odd and $m \leq d(d-3)$. We can write $\Delta = 2t+1$ for some nonnegative integer t, hence $n = d^2+2t+1$. Apply Theorem I.3(c) with $r = d^2+t$ (i.e., $r = \lfloor d\sqrt{n}\rfloor$, hence $rd(d+1)/2 \leq r^2 \leq d^2 n$). Note $\lfloor(mr + g - 1)/d\rfloor + 1 > (mr + g - 1)/d = m(d^2+t)/d + (d-3)/2$, but $m(d^2+t)/d + (d-3)/2 > m\sqrt{d^2 + 2t + 1} = m\sqrt{n}$ for $m \leq d(d-3)$. On the other hand, since $r^2 \leq d^2 n$, we see that $m\sqrt{n} \leq mnd/r$, so it suffices to show that $mnd/r \leq s+ud+1$. If $s = d-1$, then $s+ud+1 = (u+1)d = \lceil mn/r\rceil d \geq mnd/r$ as required, so assume $(s+1)(s+2) \leq 2\rho < (s+2)(s+3)$ and $s+2 \leq d$. Then $r(s+ud+1) = r(s+1)+mnd-d\rho$, so we need only check that $r(s+1)+mnd-d\rho \geq mnd$, or even that $r(s+1) \geq d(s+2)(s+3)/2$ (which is clear if $s = 0$ since $d \geq 3$) or that $r \geq d^2(s+3)/(2(s+1))$ (which is also clear since now we may assume $s \geq 1$).

Now consider (b). First assume $m \leq 2d^2/\Delta$. Let $\Delta = 2t$ and apply Theorem I.3(b) with $r = d^2 + t$, $u = m$ and $\rho = mt$; it has to be checked that $md + s + 1 \geq m\sqrt{n}$ or, equivalently, that $(s+1)^2 + 2(s+1)md \geq 2m^2t$. If $s = 0$ then $mt < 3$ and the inequality follows from $d \geq 3$, whereas if $s = d-1$ then $m \leq 2d^2/\Delta$ implies $d^2 + 2md^2 > 2m^2t$. In all intermediate cases one has $2d(s+1) \geq (s+2)(s+3)$ and $(s+2)(s+3) > 2\rho = 2mt$ which also imply $(s+1)^2 + 2(s+1)md \geq 2m^2t$ easily.

Finally, assume $m \leq d(d-3)/2$. Again $\Delta = 2t$ so $n = d^2 + 2t$; take $r = d^2 + t - 1$ and apply Theorem I.3(c) in the same manner as previously. $\diamondsuit$

Proof of Corollary I.2(a): It follows from Corollary IV.1 that $\alpha(n; m) \geq m\sqrt{n}$ holds for all $n \geq 10$ if $m \leq d(d-3)/2$, where $d = \lfloor\sqrt{n}\rfloor$ (given that $\alpha(n; m) \geq m\sqrt{n}$ is known and indeed easy to prove when n is a square). But $\sqrt{n} \geq d$ and $d^2 + 2\sqrt{n} \geq n$, so obviously $d(d-3)/2 \geq (n - 5\sqrt{n})/2$. $\diamondsuit$

V. Hilbert Functions

We now consider the problem of determining the Hilbert function of an ideal of the form $I(n;m)$. Typically Theorem I.3(b) gives a lower bound $\lambda_\alpha(n;m)$ on $\alpha(n;m)$ which is smaller than the upper bound $\Lambda_\tau(n;m)$ it gives for $\tau(n;m)$, but there are in fact many cases for which $\lambda_\alpha(n;m) \geq \Lambda_\tau(n;m)$. In any such case, it follows that $\alpha(n;m) \geq \tau(n;m)$, which clearly implies Conjecture I.1(b) for the given n and m. This is precisely the method of proof of the next result.

Corollary V.1: *Let $d \geq 3$, $\varepsilon > 0$ and $i > 0$ be integers, and consider $n = d^2 + 2\varepsilon$. Then Conjecture I.1(b) holds for the given n and m if m falls into one of the following ranges:*

(a) $(d-1)(d-2)/(2\varepsilon) \leq m < (d+2)(d+1)/(2\varepsilon)$;

(b) $(i(d^2+\varepsilon) + (d-1)(d-2)/2)/\varepsilon \leq m \leq (id^2 + d(d-1)/2)/\varepsilon$;

(c) $(i(d^2+\varepsilon) + d(d-1)/2)/\varepsilon \leq m \leq (id^2 + (d+1)d/2)/\varepsilon$; *and*

(d) $(i(d^2+\varepsilon) + (d+1)d/2)/\varepsilon \leq m \leq (id^2 + d(d+3)/2)/\varepsilon$.

Proof: Case (a) is most easily treated by considering three subcases: (a1) $(d-1)(d-2)/(2\varepsilon) \leq m < d(d-1)/(2\varepsilon)$; (a2) $d(d-1)/(2\varepsilon) \leq m < (d+1)d/(2\varepsilon)$; and (a3) $(d+1)d/(2\varepsilon) \leq m < (d+2)(d+1)/(2\varepsilon)$.

For the proof, apply Theorem I.3 with $r = d^2 + \varepsilon$, $u = m + i$ and $\rho = m\varepsilon - ir$ (with $i = 0$ for part (a)). The reader will find in cases (a1) and (b) that $s = d-3$, while $s = d-2$ in cases (a2) and (c), and $s = d-1$ in cases (a3) and (d). It follows from Theorem I.3 that $\lambda_\alpha(n;m) \geq \Lambda_\tau(n;m)$, and hence, as discussed above, Conjecture I.1(b) holds for the given n and m. $\diamondsuit$

For each n, there is a finite set of values of d, ε and i to which Corollary V.1 can be profitably applied. For example, in parts (b), (c) and (d) of Corollary V.1 we may assume $i \leq (d-1)/\varepsilon$, $i \leq d/\varepsilon$ and $i \leq d/\varepsilon$ respectively, as otherwise the corresponding range of multiplicities is empty. Thus, Corollary V.1 determines a finite set V_n of values of m for which Conjecture I.1(b) must hold. Between the least and largest m in V_n there can also be many integers m which are not in V_n. For example, of the 4200 pairs (n,m) with $10 \leq n = d^2 + 2\varepsilon \leq 100$ and $1 \leq m \leq 100$, there are 723 with $m \in V_n$. Of these, 308 have $m \leq 12$ (and thus for these Conjecture I.1(b) was verified by [CM2]); the other 415 were not known before.

It is also noteworthy that in many cases we verify Conjecture I.1(b) for quite large values of m. In particular, if $n = d^2 + 2$, it follows from Corollary V.1 that $m \in V_n$ for $m = d(d^2+1) + d(d+1)/2$. Thus we have $243 \in V_{38}$, for example, and $783 \in V_{83}$. Apart from special cases when n is a square (in particular, when n is a power of 4; see [E2]), no other method we know can handle such large multiplicities. On the other hand, as indicated by Corollary I.2(b), if n is any square larger than 10, our method also handles arbitrarily large values of m, as we now prove. For the purpose of stating the result, given any positive integer i, let l_i be the largest integer j such that $j(j+1) \leq i$.

Corollary V.2: *Consider $10 \leq n = \sigma^2$ general points of $\mathbf{P}^2$. Let k be any nonnegative integer, and let $m = x + k(\sigma - 1)$, where x is an integer satisfying $\sigma/2 - l_\sigma \leq x \leq \sigma/2$ if σ is even, or $(\sigma+1)/2 - l_{2\sigma} \leq x \leq (\sigma+1)/2$ if σ is odd. Then Corollary I.2(b) holds for $I(n;m)$.*

Proof: We apply Theorem I.3(c) with $d = \sigma - 1$, $r = d\sigma$, $u = \lceil mn/r \rceil - 1 = m + k$ and $\rho = mn - ur = x\sigma$. We claim that $t_0 \leq \min(\lfloor (mr+g-1)/d \rfloor, s+ud)$, where $t_0 = m\sigma + \sigma/2 - 2$ if σ is even and $t_0 = m\sigma + (\sigma-1)/2 - 2$ if σ is odd. But $t_0 \leq (mr+g-1)/d$ because $(mr+g-1)/d = (md\sigma + d(d-3)/2)/d = m\sigma + (\sigma-4)/2$. To see $t_0 \leq s+ud$, note that $t_0 \leq s+ud$ simplifies to $x + \sigma/2 - 2 \leq s$ if σ is even and to $x + (\sigma-1)/2 - 2 \leq s$ if σ is odd. Therefore (by definition of s) we have to check that $x + \sigma/2 - 1 \leq d$ and $(x+\sigma/2-1)(x+\sigma/2) \leq 2\sigma x$ if σ is even, and that $x + (\sigma-1)/2 - 1 \leq d$ and $(x + (\sigma-1)/2 - 1)(x + (\sigma-1)/2) \leq 2\sigma x$ if σ is odd. The first inequality follows from $x \leq \sigma/2$ and $x \leq (\sigma+1)/2$ respectively. For the second, substituting $\sigma/2 - j$ for x if σ is even and $(\sigma+1)/2 - j$ for x if σ is odd, $(x+\sigma/2-1)(x+\sigma/2) \leq 2\sigma x$ and $(x+(\sigma-1)/2-1)(x+(\sigma-1)/2) \leq 2\sigma x$ resp. become $j(j+1) \leq \sigma$ if σ is even and $j(j+1) \leq 2\sigma$ if σ is odd. Thus $(x+\sigma/2-1)(x+\sigma/2) \leq 2\sigma x$ and $(x + (\sigma-1)/2 - 1)(x + (\sigma-1)/2) \leq 2\sigma x$ resp. hold if x is an integer satisfying $\sigma/2 - l_\sigma \leq x \leq \sigma/2$ if σ is even, and $(\sigma+1)/2 - l_{2\sigma} \leq x \leq (\sigma+1)/2$ if σ is odd.

This shows by Theorem I.3(c) that $\alpha(n;m) \geq m\sigma + \sigma/2 - 1$ if σ is even and $\alpha(n;m) \geq m\sigma + (\sigma-1)/2 - 1$ if σ is odd. But since n points of multiplicity m impose at most $n\binom{m+1}{2}$ conditions on forms of degree t, it follows that $h(n;m)(t) \geq \binom{t+2}{2} - n\binom{m+1}{2}$, and it is easy to check that $\binom{t+2}{2} - n\binom{m+1}{2} > 0$ whenever $t \geq m\sigma + \sigma/2 - 1$ if σ is even and $t \geq m\sigma + (\sigma-1)/2 - 1$ if σ is odd. Thus in fact we have $\alpha(n;m) = m\sigma + \sigma/2 - 1$ if σ is even and $\alpha(n;m) = m\sigma + (\sigma-1)/2 - 1$ if σ is odd, whenever m is of the form $m = x + k(\sigma - 1)$, with x as given in the statement of Corollary V.2.

Of course, $h(n;m)(t) = 0$ for all $t < \alpha(n;m)$, and by [HHF], we know that $h(n;m)(t) = \binom{t+2}{2} - n\binom{m+1}{2}$ for all $t \geq \alpha(n;m)$ (apply Lemma 5.3 of [HHF], keeping in mind our explicit expression for $\alpha(n;m)$). $\diamondsuit$

Note that Corollary I.2(b) is an immediate consequence of the preceding result.

VI. Resolutions

We now show how our results verify many cases of Conjecture I.1(c) also, including cases with m arbitrarily large. Indeed, in addition to the case that n is an even square treated in Corollary VI.2 below, we have by the following proposition the resolution for 121 of the 723 pairs (n,m) with $m \in V_n$ mentioned above, and of these 121, 91 have $m > 2$ and hence were not known before.

Corollary VI.1: *Let $d \geq 3$ and $\varepsilon \geq 1$ be integers. Then Conjecture I.1(c) holds for each of the following values of n and m, whenever m is an integer:*

(a) *$n = d^2 + 2\varepsilon$ and $m = d(d \pm 1)/(2\varepsilon)$, in which case $\alpha = \alpha(n;m) = md + d - 1/2 \pm 1/2$ and the minimal free resolution of $I(n;m)$ is*

$$0 \to R[-\alpha-1]^{\alpha} \to R[-\alpha]^{\alpha+1} \to I(n;m) \to 0;$$

(b) *$n = d^2 + 2\varepsilon$ and $m = (d(d \pm 1)/2 - 1)/\varepsilon$, in which case $\alpha = \alpha(n;m) = md + d - 3/2 \pm 1/2$ and the minimal free resolution of $I(n;m)$ is*

$$0 \to R[-\alpha-2]^{b+m} \to R[-\alpha-1]^{b} \oplus R[-\alpha]^{m+1} \to I(n;m) \to 0,$$

where $b = (m+1)(d-2) + 1/2 \pm 1/2$;

(c) $n = d^2+2$ *and* $m = d^2 + d(d \pm 1)/2$, *in which case* $\alpha = \alpha(n;m) = (m+1)d + d - 3/2 \pm 1/2$ *and the minimal free resolution of* $I(n;m)$ *is*

$$0 \to R[-\alpha-2]^{a+b-1} \to R[-\alpha-1]^b \oplus R[-\alpha]^a \to I(n;m) \to 0,$$

where $a = d(d\pm 1)/2$ *and* $b = \alpha + 2 - d(d \pm 1)$.

Proof: For case (a), apply Corollary V.1(a2) for $m = d(d-1)/(2\varepsilon)$ and Corollary V.1(a3) for $m = d(d+1)/(2\varepsilon)$. It turns out that $\alpha(n;m) > \tau(n;m)$ in these cases, but it is well known that $I(n;m)$ is generated in degrees $\tau(n;m)+1$ and less, hence in degree $\alpha(n;m)$, from which it follows (see the displayed formula following Definition 2.4 of [HHF]) that the minimal free resolution is as claimed.

For cases (b) and (c), it turns out that $\alpha(n;m) = \tau(n;m)$: for case (b), apply Corollary V.1(a1-2), resp., while for case (c), apply Corollary V.1(b,c), resp., using $i = \varepsilon = 1$. To obtain the resolution, consider $\mathbf{m} = (m_1, \ldots, m_n)$, where $m_1 = m+1$ and $m_2 = \cdots = m_n = m$, and apply Theorem I.3 to $\alpha(\mathbf{m})$ using $r = d^2 + \varepsilon$. It turns out that $\alpha(\mathbf{m}) > \alpha(n;m)$. Now by Lemma 2.6(b) of [HHF] it follows that Conjecture I.1(c) holds and that the minimal free resolutions are as claimed (again, see the displayed formula following Definition 2.4 of [HHF]). $\diamondsuit$

When n is an even square, Corollary V.2, together with Theorem 5.1(a) of [HHF], directly implies:

Corollary VI.2: *Consider* $n = \sigma^2$ *general points of* $\mathbf{P}^2$, *where* $\sigma > 3$ *is even. Let* k *be any nonnegative integer, and let* $m = x + k(\sigma - 1)$, *where* x *is an integer satisfying* $\sigma/2 - l_\sigma \leq x \leq \sigma/2$. *Then Conjecture I.1(c) holds for* $I(n;m)$.

Note that Corollary I.2(c) is an immediate consequence of the preceding result.

VII. Comparisons

It is interesting to carry out some comparisons. First we compare our results here with each other. Sometimes the best bound determined by Theorem I.3 comes from parts (a) or (c), and sometimes it comes from part (b). Sometimes, of course, one can do better applying our algorithm for values of r and d for which Theorem I.3 does not apply.

For example, let $\alpha_c(n;m)$ denote the conjectural value of $\alpha(n;m)$ and let $\tau_c(n;m)$ denote the conjectural value of $\tau(n;m)$ (i.e., the values of each assuming Conjecture I.1(b) holds). Then $\alpha_c(33;29) = 168$; the best bound given by Theorem I.3 is $\alpha(33;29) \geq 165$, obtained in part (b) using $r = 29$ and $d = 5$, or in part (c) using $r = 17$ and $d = 3$. Applying our algorithm with $r = 23$ and $d = 4$, however, gives $\alpha(33;29) \geq 168$ (and hence $\alpha(33;29) = \alpha_c(33;29)$). On the other hand, $\alpha_c(38;16) = 101$ and indeed we obtain $\alpha(38;16) \geq 101$ via Theorem I.3(b) using $r = 37$ and $d = 6$, while the best bound obtainable via Theorem I.3(c) is $\alpha(38;16) \geq 98$, gotten using $r = 36$ and $d = 6$. In contrast, we obtain $\alpha(119;13) \geq 146 = \alpha_c(119;13)$ via Theorem I.3(c) using $r = 109$ and $d = 10$, while the best bound obtainable via Theorem I.3(b) is $\alpha(119;13) \geq 144$, gotten using $r = 100$ and $d = 9$.

Similarly, $\tau_c(33;29) = 168$; applying our algorithm with $r = 23$ and $d = 4$, gives $\tau(33;29) \leq 169$. The best bound given by Theorem I.3(b) is $\tau(33;29) \leq 170$, obtained using $r = 29$ and $d = 5$, while the best bound given by Theorem I.3(a) is $\tau(33; 29) \leq 175$, obtained using $r = 33$ and $d = 6$. On the other hand, $\tau_c(38; 16) = 101$ and indeed we obtain $\tau(38; 16) \leq 101$ via Theorem I.3(b) using $r = 37$ and $d = 6$, while the best bound obtainable via Theorem I.3(c) is $\tau(38; 16) \leq 103$, gotten using $r = 36$ and $d = 6$. In contrast, $\tau_c(119; 13) = 146$, and the best bound obtainable using our algorithm is $\tau(119; 13) \leq 147$, obtained using $r = 119$ and $d = 11$ (and hence Theorem I.3(a) applies), while the best bound obtainable via Theorem I.3(b) is $\tau(119; 13) \leq 148$, gotten using $r = 111$ and $d = 10$.

We now compare our bounds with previously known bounds. Suppose that $rd(d+1)/2 \leq r^2 < nd^2$; then for m large enough the bound from Theorem I.3(c) is $\alpha(n; m) \geq 1 + \lfloor (mr + g - 1)/d \rfloor$. This is better than the bound of Corollary IV.1.1.2 of [H4] (which generalizes the main theorem of [H3]; see [H5] for further generalizations and related results), which is just $\alpha(n; m) \geq mr/d$. On the other hand, suppose that $2r \geq n + d^2$. Then $r^2 \geq nd^2$ (because the arithmetic mean is never less than the geometric mean), so the main theorem of [H3] applies and gives $\alpha(n; m) \geq mnd/r$. Typically Theorem I.3(b) gives a better bound than this, but if in addition r divides mn, the bound from Theorem I.3(b) simplifies, also giving $\alpha(n; m) \geq mnd/r$.

In fact, for given r and d, the bound in [H3], and the generalization given in Theorem IV.1.1.1 of [H4], can be shown (see [H4]) never to give a better bound on α than that given by the algorithms of Section II. When m is large enough compared with n, [H3] shows its bound on $\alpha(n; m)$ is better than those of [R1], and thus so are the bounds here.

When m is not too large compared with n, the bounds on α given by Theorem I.3, like the bound given by the unloading algorithm of [R1], are among the few that sometimes give bounds better than the bound $\alpha(n; m) \geq \lfloor m\sqrt{n} \rfloor + 1$ conjectured in [N1]. Consider, for example, $n = 1000$ and $m = 13$: [R1] gives $\alpha(n; m) \geq 421$ and Theorem I.3, using $r = 981$ and $d = 31$, gives $\alpha(n; m) \geq 424$, whereas $\lfloor m\sqrt{n} \rfloor + 1 = 412$; $\alpha_c(n; m)$ is 426 in this case. See Corollary IV.1 for more examples. Moreover, Theorem I.3 is the only result that we know which sometimes determines $\alpha(n; m)$ exactly even for m reasonably large compared to n, when n is not a square.

Here are some comparisons for τ. Bounds on $\tau(n; m)$ given by Hirschowitz [Hi1], Gimigliano [Gi] and Catalisano [C1] are on the order of $m\sqrt{2n}$. Thus, for sufficiently large m, the bound $\tau(n; m) \leq m\lceil \sqrt{n} \rceil + \lceil (\lceil \sqrt{n} \rceil - 3)/2 \rceil$ given in [HHF] for $n > 9$ is better. In fact, [HHF] shows that $\tau(n; m) \leq m\lceil \sqrt{n} \rceil + \lceil (\lceil \sqrt{n} \rceil - 3)/2 \rceil$ is an equality when $n > 9$ is a square and m is sufficiently large. However, when n is a square, Theorem I.3(a), using $d^2 = r = n$, also gives this bound (this is to be expected, since the method we use is based on the method used in [HHF]), and when n is not a square, Theorem I.3(a), using $d = \lceil \sqrt{n} \rceil$ and $r = n$, gives a bound that is less than or equal to that of [HHF] (although never more than 2 smaller). But one can also apply Theorem I.3(b) using other values of r and d, and often do much better. In addition, as was pointed out for α above, Theorem I.3 is the only result that we know that sometimes determines $\tau(n; m)$ exactly for values of m and n that can be large, even when n is not a square.

Other bounds on τ have also been given. Bounds given by Xu [X] and Ballico [B] are on the order of $m\sqrt{n}$, but nonethless the bound from [HHF] (and hence Theorem I.3) is better than Xu's when n is large enough and better than Ballico's when m is large enough.

For large m, the bound given in [R2] is also better than those of [B] and [X], and by an argument similar to the one used in [H3] to compare the bounds on α, the bounds here on $\tau(n;m)$ are better than those of [R2] when m is large enough compared with n.

For example, for $n = 190$ and $m = 100$, then $\tau_c(n;m) = 1384$, while Theorem I.3(b), using $r = 180$ and $d = 13$, gives $\tau(n;m) \leq 1390$, and we have in addition:

- $\tau(Z) \leq 1957$ from [Hi1],
- $\tau(Z) \leq 1900$ from [Gi],
- $\tau(Z) \leq 1899$ from [C1],
- $\tau(Z) \leq 1487$ from [B],
- $\tau(Z) \leq 1465$ from [X],
- $\tau(Z) \leq 1440$ from [R2] and
- $\tau(Z) \leq 1406$ from [HHF].

For a different perspective, we close with some graphs which show our results, and what was known previously. For Figure 1, the references are: [N2] for Nagata, [Hi2] for Hirschowitz, [CM1] and [CM2] for Ciliberto and Miranda, and [E2] for Evain. The data shown in Figure 2 is simply a graphical representation of Corollary V.1 and Corollary V.2.

For Figure 3, the references are: [GGR] for Geramita, Gregory and Roberts, [C2] for Catalisano, [H2] for Harbourne, [I] for Idà, and [HHF] for Harbourne, Holay, and Fitchett. The data shown in Figure 4 is simply a graphical representation of Corollary VI.1 and Corollary VI.2.

Finally, Figure 5 is a graph of all (n, m) for which Theorem I.3, using $d = \lfloor\sqrt{n}\rfloor$ and $r = \lfloor(n + d^2)/2\rfloor$, implies Conjecture I.1(a). It turns out that this occurs for 131261, or 69.8%, of the $470 \cdot 400 = 188000$ points in the range $10 \leq n \leq 480$, $1 \leq m \leq 400$, shown on the graph.

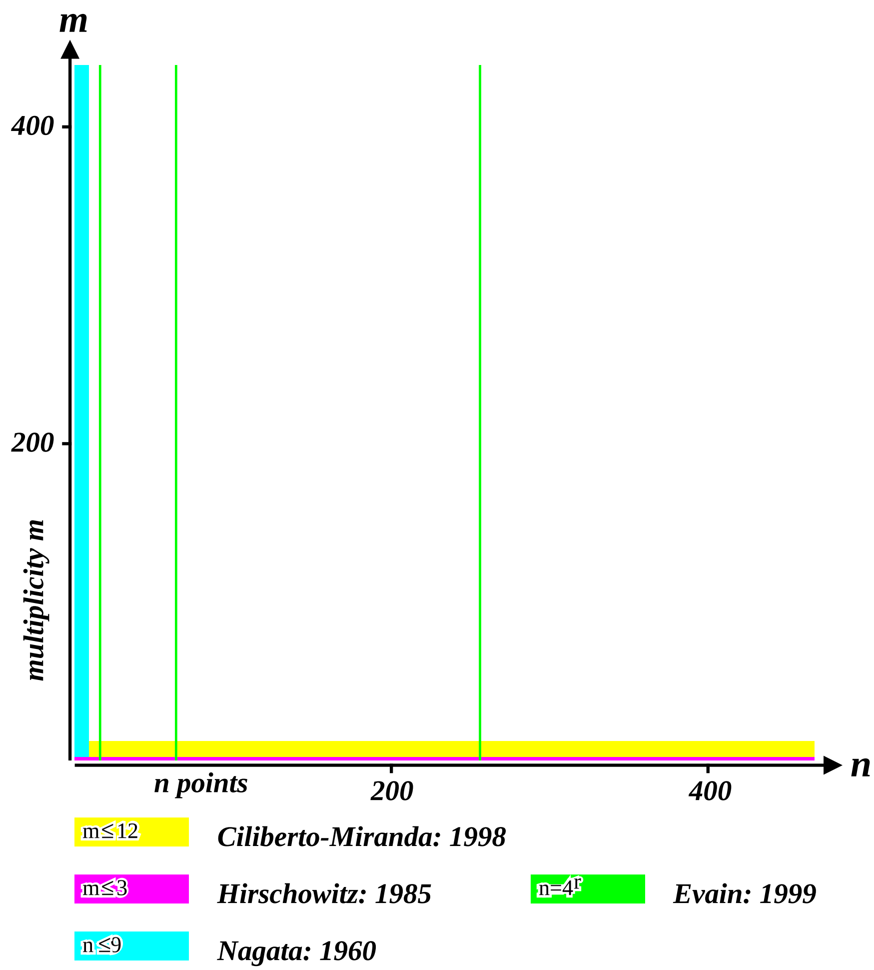


Figure 1: Graph showing for which (n,m) the Hilbert function of the ideal I(n;m) of n general points in $\mathbf{C}P^2$ of multiplicity m is known, and by whom, as of 1999.

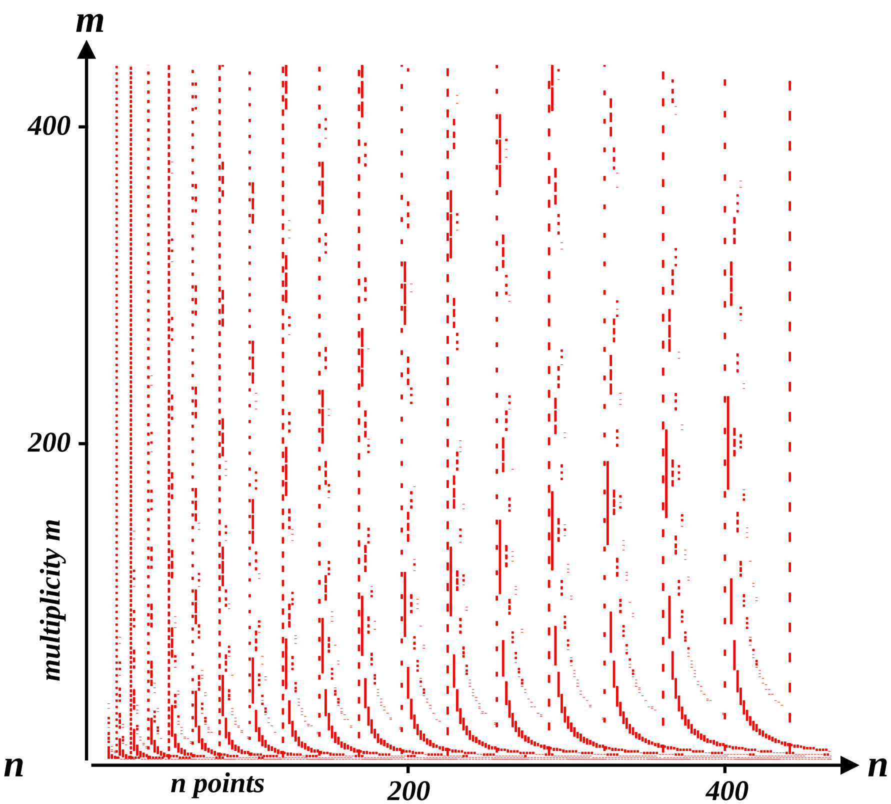


Figure 2: Graph showing for which (n,m) the Hilbert function of the ideal I(n;m) of n general points in $\mathbf{C}P^2$ of multiplicity m is known via Corollaries V.1, V.2.

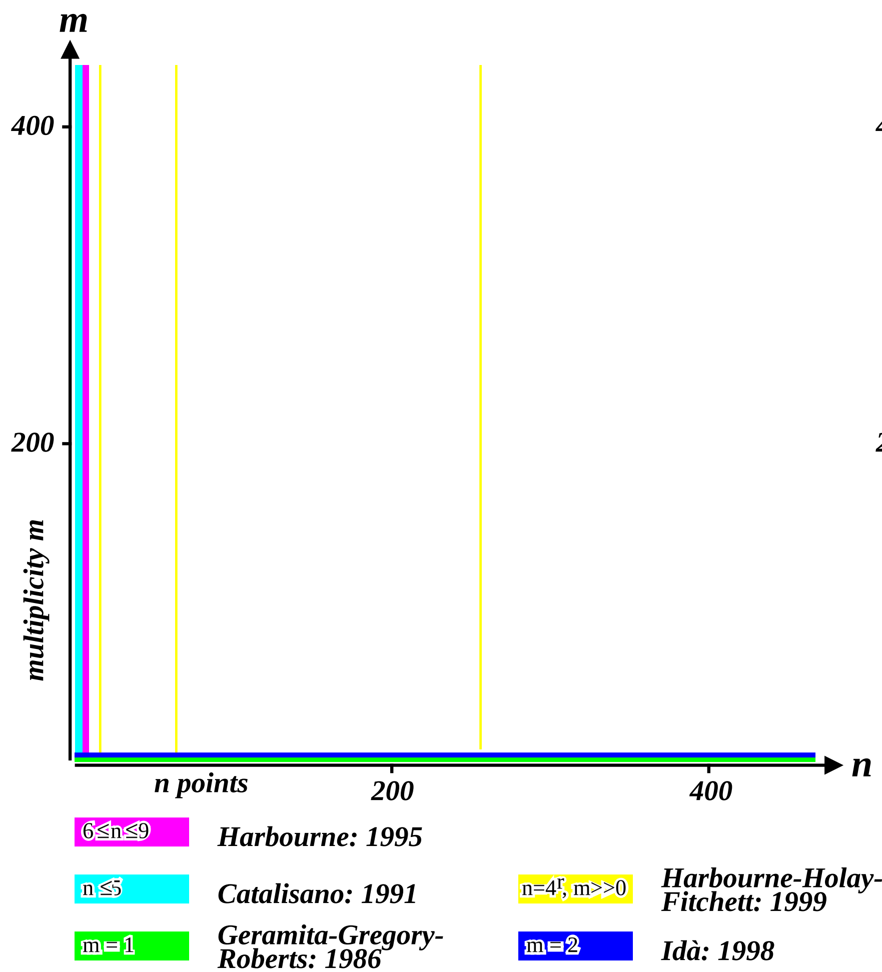


Figure 3: Graph showing for which (n,m) the resolution of the ideal I(n;m) of n general points in $\mathbf{C}P^2$ of multiplicity m is known, and by whom, as of 1999.

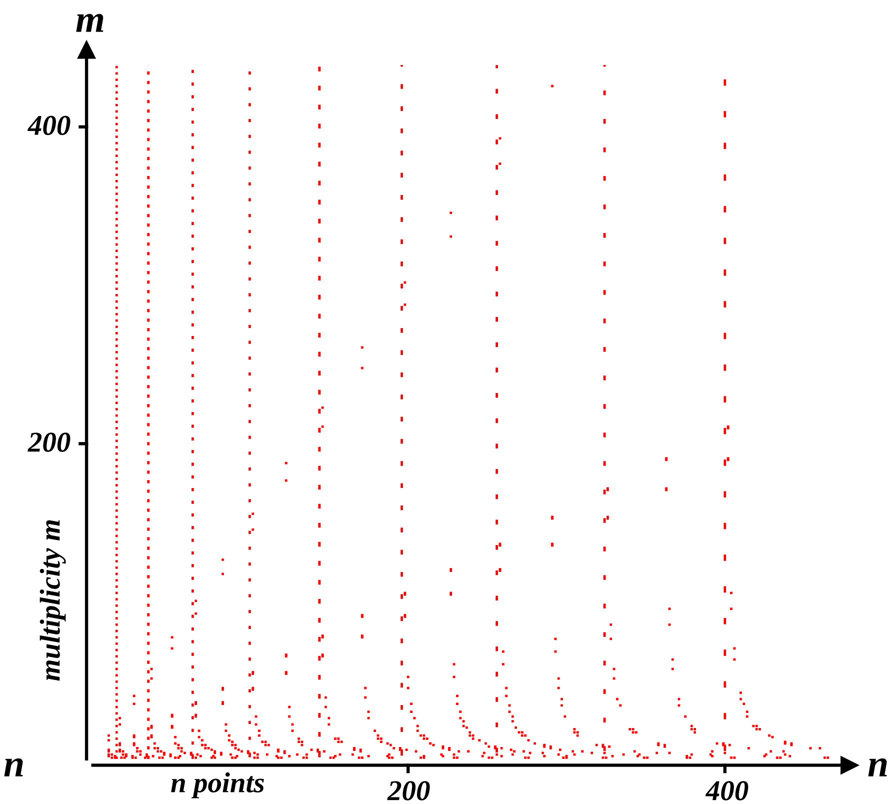
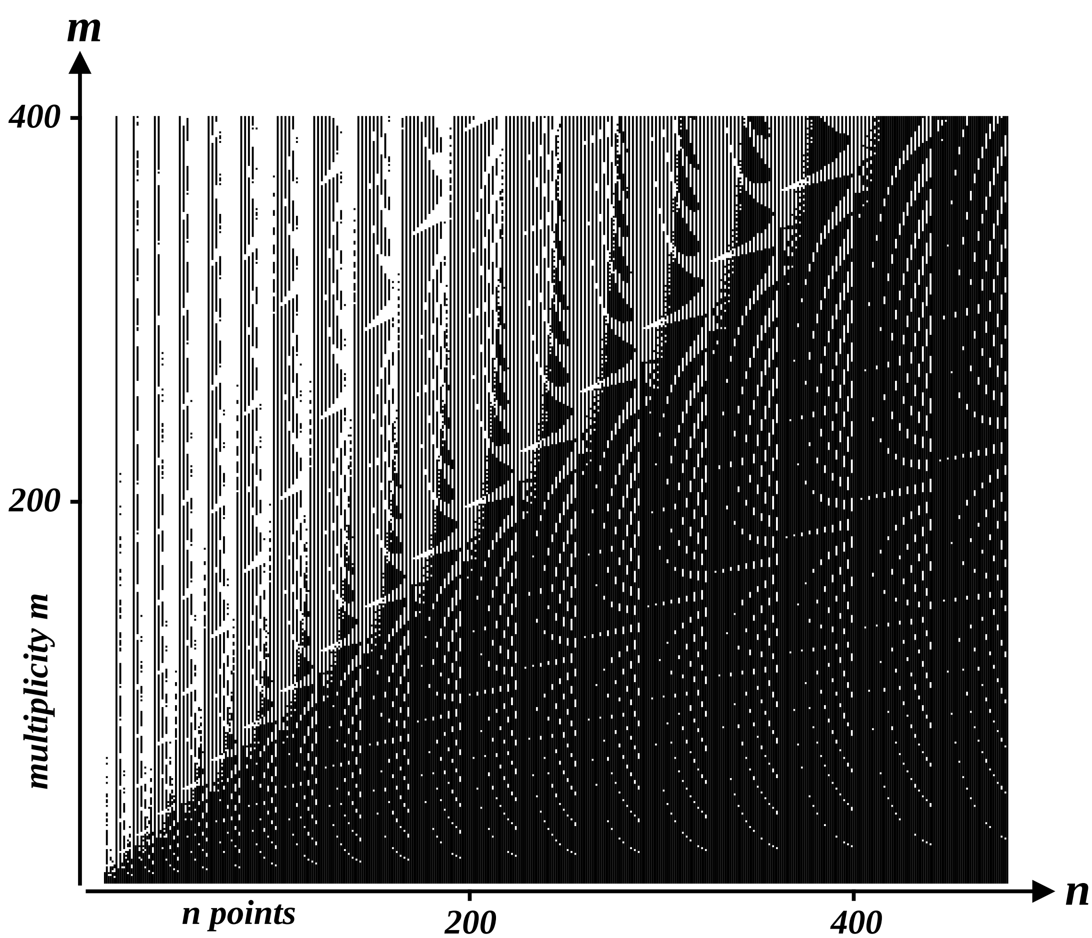


Figure 4: Graph showing for which (n,m) the resolution of the ideal I(n;m) of n general points in $\mathbf{C}P^2$ of multiplicity m is known via Corollaries VI.1, VI.2.



Graph showing all (n, m) such that the Harbourne-Roé bounds, using $d = \lfloor\sqrt{n}\rfloor$ and $r = \lfloor(n + d^2)/2\rfloor$, imply Nagata's conjectural bound $\alpha(n; m) \geq m\sqrt{n}$.

Figure 5

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