

Homework 3: Math 953 Spring 2005

Due February 7, 2005

(1) Let \mathbf{R} have the usual topology. Let \mathcal{P} be the presheaf of bounded locally constant continuous functions on \mathbf{R} . I.e., for each open set $U \subset \mathbf{R}$, let $\mathcal{P}(U)$ be the set of all continuous functions $f : U \rightarrow \mathbf{R}$ such that for each f and U there is an M with $|f(x) - f(y)| \leq M$ for all x and y in U , and such that for each $x \in U$, U contains a neighborhood V of x such that $f(y) = f(x)$ for all $y \in V$.

(a) Show that evaluation defines an isomorphism $\mathcal{P}_p \rightarrow \mathbf{R}$ for each stalk \mathcal{P}_p of \mathcal{P} .

(b) Show that \mathcal{P} is not a sheaf.

(c) Let \mathcal{L} be the presheaf of locally constant continuous functions; this is defined the same way as \mathcal{P} but without requiring boundedness. Show that \mathcal{L} is a sheaf, and define a morphism $\mathcal{P} \rightarrow \mathcal{L}$ which induces an isomorphism on stalks. Conclude that \mathcal{L} is (isomorphic to) the sheafification of \mathcal{P} .

(2) There are various versions of the Nullstellensatz. Here are four. (If we impose the condition in (c) and (d) that k is algebraically closed, then all four are equivalent.) Pick one and show that it implies one of the others.

(a) (Weak Nullstellensatz) Let $R = k[x_1, \dots, x_n]$ be the polynomial ring in n variables over an algebraically closed field k , and let $J \subset R$ be an ideal. If $Z(J) \subset k^n$ is empty, then $J = R$. (When $n = 1$, this just says that any nonconstant polynomial has a root in k .)

(b) (Strong Nullstellensatz) Let $R = k[x_1, \dots, x_n]$ be the polynomial ring in n variables over an algebraically closed field k , and let $J \subset R$ be an ideal. Then $I(Z(J)) = \sqrt{J}$.

(c) (Ring version) Let $R = k[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field k , and let $J \subset R$ be an ideal. Let I be the intersection of all maximal ideals that contain J . Then $I = \sqrt{J}$.

(d) (Field version) Let $R = k[x_1, \dots, x_n]$ be the polynomial ring in n variables over a field k , and let $J \subset R$ be an ideal such that $K = R/J$ is a field. Then K is an algebraic extension of k .

(3) Let $f_1 : A_1 \rightarrow A_3$ and $f_2 : A_2 \rightarrow A_3$ be maps of sets. Regarding f_1 and f_2 as giving an inverse system, $\varprojlim A_i$ turns out to be the fiber product, denoted $A_1 \times_{A_3} A_2$. In the category of sets, it is just $\{(a_1, a_2) \in A_1 \times A_2 : f_1(a_1) = f_2(a_2)\}$. (See Lang, for example, for fiber products, pullbacks, and inverse limits.)

(a) Let A_i for $i = 1, 2$ both be the parabola $y = x^2$ in the plane. Let f_1 be the projection to the x -axis and let f_2 be the projection to the y -axis. Determine $A_1 \times_{A_3} A_2$ as a subset of the usual product $A_1 \times A_2$.

(b) We can also regard f_i as a direct system. If so, what do we get for $\varinjlim A_i$?

(4) Let $f_2 : A_1 \rightarrow A_2$ and $f_3 : A_1 \rightarrow A_3$ be homomorphisms of rings. The direct limit $\varinjlim A_i$ in this case is just the tensor product $A_2 \otimes_{A_1} A_3$. (See Lang, for example, for tensor products, pushouts, and direct limits.)

(a) Let $f_i : \mathbf{Z} \rightarrow \mathbf{Z}/3^i\mathbf{Z}$, for $i = 2, 3$. Show that $\varinjlim A_i$ is isomorphic to $\mathbf{Z}/m\mathbf{Z}$ for some m and determine m (using either what you know about tensor products, or showing $\mathbf{Z}/m\mathbf{Z}$ satisfies the universal property of direct limits.)

(b) We can also regard f_i here as an inverse system. If so, what do we get for $\varprojlim A_i$?