M417 Homework 3 Solutions Spring 2004

(1) (a) For any subsets $C_1, C_2 \subset A$, show that $f(C_1 \cup C_2) = f(C_1) \cup f(C_2)$: We must show that any element of $f(C_1 \cup C_2)$ is an element of $f(C_1) \cup f(C_2)$, and vice versa. So let $y \in f(C_1 \cup C_2)$. Then $y = f(x)$ for some $x \in C_1 \cup C_2$. If $x \in C_1$, then $y \in f(C_1) \subset f(C_1 \cup C_2)$, and if $x \in C_2$, then $y \in f(C_2) \subset f(C_1 \cup C_2)$. This shows that $f(C_1 \cup C_2) \subset f(C_1) \cup f(C_2)$. To see that $f(C_1) \cup f(C_2) \subset f(C_1 \cup C_2)$, let $y \in f(C_1) \cup f(C_2)$. Then either $y \in f(C_1) \subset f(C_1 \cup C_2)$ or $y \in f(C_2) \subset f(C_1 \cup C_2)$. Hence $y \in f(C_1 \cup C_2)$, so $f(C_1) \cup f(C_2) \subset f(C_1 \cup C_2)$. Since each of $f(C_1) \cup f(C_2)$ and $f(C_1 \cup C_2)$ contains the other, we have $f(C_1) \cup f(C_2) = f(C_1 \cup C_2)$.

Here's an alternative proof: $f(C_1 \cup C_2) = \{f(x) : x \in C_1 \cup C_2\} = \{f(x) : x \in C_1\} \cup \{f(x) : x \in C_2\} = f(C_1) \cup f(C_2)$.

(b) For any subsets $D_1, D_2 \subset B$, show that $f^{-1}(D_1 \cup D_2) = f^{-1}(D_1) \cup f^{-1}(D_2)$: if $x \in f^{-1}(D_1 \cup D_2)$, then $f(x) \in D_1 \cup D_2$, so either $f(x) \in D_1$ (and hence $x \in f^{-1}(D_1)$), or $f(x) \in D_2$ (and hence $x \in f^{-1}(D_2)$). Thus $x \in f^{-1}(D_1) \cup f^{-1}(D_2)$, so $f^{-1}(D_1 \cup D_2) \subset f^{-1}(D_1) \cup f^{-1}(D_2)$. On the other hand, if $x \in f^{-1}(D_1) \cup f^{-1}(D_2)$, then either $x \in f^{-1}(D_1)$ (and hence $f(x) \in D_1 \subset D_1 \cup D_2$) or $x \in f^{-1}(D_2)$ (and hence $f(x) \in D_2 \subset D_1 \cup D_2$). Thus $x \in f^{-1}(D_1 \cup D_2)$, so $f^{-1}(D_1) \cup f^{-1}(D_2) \subset f^{-1}(D_1 \cup D_2)$. Thus each of $f^{-1}(D_1) \cup f^{-1}(D_2)$ and $f^{-1}(D_1 \cup D_2)$ contains the other, so $f^{-1}(D_1) \cup f^{-1}(D_2) = f^{-1}(D_1 \cup D_2)$.

Here's an alternative proof: $f^{-1}(D_1 \cup D_2) = \{x : f(x) \in D_1 \cup D_2\} = \{x : f(x) \in D_1\} \cup \{x : f(x) \in D_2\} = f^{-1}(D_1) \cup f^{-1}(D_2)$.

(2) For any subsets $C_1, C_2 \subset A$, show that $f(C_1 \cap C_2) \subset f(C_1) \cap f(C_2)$. Give an example to show that $f(C_1 \cap C_2) = f(C_1) \cap f(C_2)$ can fail: sets C_1 and C_2 can be found to make this fail whenever f is not injective. Say $a \neq b$, but $f(a) = f(b)$. Take $C_1 = \{a\}$, $C_2 = \{b\}$. Then $f(C_1 \cap C_2) = f(\emptyset) = \emptyset$, but $f(C_1) \cap f(C_2) = \{f(a)\}$. To see that $f(C_1 \cap C_2) \subset f(C_1) \cap f(C_2)$, let $y \in f(C_1 \cap C_2)$. Then $y = f(x)$ for some $x \in C_1 \cap C_2$. Since x is in both C_1 and C_2, y is in both $f(C_1)$ and $f(C_2)$, so $y \in f(C_1) \cap f(C_2)$, hence $f(C_1 \cap C_2) \subset f(C_1) \cap f(C_2)$.

Here's an alternative proof: Since $C_1 \cap C_2 \subset C_1$, certainly, $f(C_1 \cap C_2) \subset f(C_1)$. Similarly, $f(C_1 \cap C_2) \subset f(C_2)$, so $f(C_1 \cap C_2) \subset f(C_1) \cap f(C_2)$, as we wanted to show.

(3) For any subsets $D_1, D_2 \subset B$, show that $f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)$: if $x \in f^{-1}(D_1 \cap D_2)$, then $f(x) \in D_1 \cap D_2$, so $f(x)$ is in both D_1 and D_2, hence x is in both $f^{-1}(D_1)$ and $f^{-1}(D_2)$, so $x \in f^{-1}(D_1) \cap f^{-1}(D_2)$, which shows $f^{-1}(D_1 \cap D_2) \subset f^{-1}(D_1) \cap f^{-1}(D_2)$. To see $f^{-1}(D_1) \cap f^{-1}(D_2) \subset f^{-1}(D_1 \cap D_2)$, let $x \in f^{-1}(D_1) \cap f^{-1}(D_2)$. Then $f(x)$ is in both D_1 and D_2; i.e., $f(x) \in D_1 \cap D_2$, so $x \in f^{-1}(D_1 \cap D_2)$. Thus $f^{-1}(D_1) \cap f^{-1}(D_2) \subset f^{-1}(D_1 \cap D_2)$, hence $f^{-1}(D_1 \cap D_2) = f^{-1}(D_1) \cap f^{-1}(D_2)$.

Here's an alternative proof: $f^{-1}(D_1 \cap D_2) = \{x : f(x) \in D_1 \cap D_2\} = \{x : f(x) \in D_1\} \cap \{x : f(x) \in D_2\} = f^{-1}(D_1) \cap f^{-1}(D_2)$.

(4) Show that $C \subset f^{-1}(f(C))$ for every subset $C \subset A$, and that equality always holds if and only if f is injective: let $x \in C$. Then $y = f(x) \in f(C)$, so $x \in f^{-1}(f(C))$, hence $C \subset f^{-1}(f(C))$.

Now we show that $C = f^{-1}(f(C))$ for every subset $C \subset A$ if and only if f is injective: If f is not injective, then we can find $a \neq b$ where $f(a) = f(b)$, hence $\{a, b\} \subset f^{-1}(f(\{a\}))$, so $C = f^{-1}(f(C))$ fails for $C = \{a\}$. This shows that if $C = f^{-1}(f(C))$ always holds, then f is injective. Conversely, if f is injective, let $x \in f^{-1}(f(C))$. Then $f(x) \in f(C)$, so $f(x) = f(a)$ for some $a \in C$. Hence $x = a \in C$, which means $f^{-1}(f(C)) \subset C$. We already know that $C \subset f^{-1}(f(C))$, so this means $C = f^{-1}(f(C))$.

(5) Show that $f(f^{-1}(D)) \subset D$ for every subset $D \subset B$, and that equality always holds if and only if f is surjective: let $y \in f(f^{-1}(D))$. Then $y = f(x)$ for some $x \in f^{-1}(D)$, so $y = f(x) \in D$, which shows that $f(f^{-1}(D)) \subset D$.

Now we show that $f(f^{-1}(D)) = D$ for every subset $D \subset B$ if and only if f is surjective: If f is not surjective, then there is a $y \in B - f(A)$. Taking $D = \{y\}$, we see that $f^{-1}(D) = \emptyset$, hence $f(f^{-1}(D)) = \emptyset \neq D$. This shows that if $f(f^{-1}(D)) = D$ always holds, then f must be surjective. Now suppose f is surjective. Let $y \in D$. Surjectivity guarantees $y = f(x)$ for some x, hence $x \in f^{-1}(D)$, so $y = f(x) \in f(f^{-1}(D))$, which shows that $D \subset f(f^{-1}(D))$, and hence $f(f^{-1}(D)) = D$.