Notes on Complete Ideals on the Plane

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While the results of this note are known, there seems to be no written account taking the biregular point of view we present here (as opposed to the usual local and birational point of view). Since our interest in complete ideals is related to making sense of fat point subschemes involving possibly infinitely near points, we begin by discussing how to use complete ideals to extend the usual treatment of fat points.

A fat point subscheme $Z = m_1p_1 + \cdots + m_rp_r$ usually is considered in the case that the points $\{p_i\}$ are distinct points. For example, let $\pi : X \to \mathbf{P}^2$ be the birational morphism obtained by blowing up distinct points $p_1, \ldots, p_r$ of $\mathbf{P}^2$. Given nonnegative integers m_i and the fat point subscheme $Z = m_1p_1 + \cdots + m_rp_r$, let $\mathcal{I}_Z$ be the sheaf of ideals defining Z as a subscheme of $\mathbf{P}^2$. Let e_0 be the pullback to X of the class of a line on $\mathbf{P}^2$, and let $e_1, \ldots, e_r$ be the classes of the exceptional divisors of the blow ups of $p_1, \ldots, p_r$. Given a divisor class f we will denote the corresponding line bundle by $\mathcal{O}_X(f)$. With this convention, then $\mathcal{I}_Z = \pi_*(\mathcal{O}_X(-m_1e_1 - \cdots - m_re_r))$ and the stalks of $\mathcal{I}_Z$ are complete ideals (as defined in [Z] and [ZS]) in the local rings of the structure sheaf of $\mathbf{P}^2$.

However, the assumption that the points are distinct is not necessary. In particular, let $p_1 \in X_0 = \mathbf{P}^2$, and let $p_2 \in X_1, \ldots, p_r \in X_{r-1}$, where, for $0 \leq i \leq r-1$, $\pi_i : X_{i+1} \to X_i$ is the blow up of p_{i+1}. We will denote X_r by X and the composition $X \to \mathbf{P}^2$ by π. We call the points $p_1, \ldots, p_r$ *essentially distinct* points of $\mathbf{P}^2$; note that p_j for $j > i$ may be infinitely near p_i. Denoting the class of the 1-dimensional scheme-theoretic fiber E_i of $X_r \to X_i$ by e_i and the pullback to $X = X_r$ of the class of a line in $\mathbf{P}^2$ by e_0, we have what we call the *exceptional configuration* $e_0, \ldots, e_r$ corresponding to $p_1, \ldots, p_r$. Then $\pi_*(\mathcal{O}_X(-m_1e_1 - \cdots - m_re_r))$ is a coherent sheaf of ideals on $\mathbf{P}^2$ defining a 0-dimensional subscheme Z generalizing the usual notion of fat point subscheme. In analogy with the notation used above, we will denote Z by $m_1p_1 + \cdots + m_rp_r$ and refer to Z as a fat point subscheme. Moreover, the stalks of $\pi_*(\mathcal{O}_X(-m_1e_1 - \ldots - m_re_r))$ are complete ideals in the stalks of the local rings of the structure sheaf of $\mathbf{P}^2$, and conversely if $\mathcal{I}$ is a coherent sheaf of ideals on $\mathbf{P}^2$ whose stalks are complete ideals and if $\mathcal{I}$ defines a 0-dimensional subscheme, then there are essentially distinct points $p_1, \ldots, p_r$ of $\mathbf{P}^2$ and integers m_i such that with respect to the corresponding exceptional configuration we have $\mathcal{I} = \pi_*(\mathcal{O}_X(-m_1e_1 - \cdots - m_re_r))$. Thus our generalized notion of fat points is precisely what is obtained by considering 0-dimensional subschemes defined by coherent sheaves of ideals whose stalks are complete ideals (see Remark 18); justifying this statement is the main purpose of this appendix.

The subscheme Z does not uniquely determine $-m_1e_1 - \cdots - m_re_r$. For example, if p_1 and p_2 are distinct points of $\mathbf{P}^2$, then $\pi_*(\mathcal{O}_X(-e_1 + e_2)) = \pi_*(\mathcal{O}_X(-e_1))$ both give the sheaf of ideals defining the subscheme $Z = p_1$. To get uniqueness, we recall that the divisor class group $\mathrm{Cl}(X)$ supports an intersection form, with respect to which the exceptional configuration $e_0, \ldots, e_r$ is an orthogonal basis of $\mathrm{Cl}(X)$ with $-1 = -e_0^2 = e_1^2 = \cdots = e_r^2$. The inequalities $(-m_1e_1 - \cdots - m_re_r) \cdot C_{ij} \geq 0$, where i indexes the divisors E_i and j indexes the components C_{ij} of E_i, correspond to what older terminology called the *proximity inequalities*. Thus we will say that a class $f \in \mathrm{Cl}(X)$ satisfies the proximity inequalities if $f \cdot C \geq 0$ for every component C of each divisor E_i. Moreover, given essentially distinct points $p_1, \ldots, p_r$ and a subscheme $Z = m_1p_1 + \cdots + m_rp_r$, we will abbreviate saying that the class $-m_1e_1 - \cdots - m_re_r$ coming from the coefficients $m_1, \ldots, m_r$ used to define Z satisfies the proximity inequalities by simply saying that Z satisfies the proximity inequalities. In case $p_1, \ldots, p_r$ are distinct points, we note that $m_1p_1 + \cdots + m_rp_r$ satisfies the proximity inequalities if and only if $m_i \geq 0$ for all i.

From one point of view, the significance of the proximity inequalities is given by an old and well-known result saying that $de_0 - m_1e_1 - \cdots - m_re_r$ is fixed component free for d sufficiently large if and only if $-m_1e_1 - \cdots - m_re_r$ satisfies the proximity inequalities. Another manifestation of the proximity inequalities is the fact that if $\pi_*(\mathcal{O}_X(-a_1e_1 - \cdots - a_re_r)) = \pi_*(\mathcal{O}_X(-b_1e_1 - \cdots - b_re_r))$, where $-a_1e_1 - \cdots - a_re_r$ and

$-b_1e_1 - \cdots - b_re_r$ both satisfy the proximity inequalities, then $a_i = b_i$ for each i. In particular, we have a bijection between subschemes of generalized fat points in $\mathbf{P}^2$ and 0-cycles $m_1p_1+\cdots+m_rp_r$, where $p_1, \ldots, p_r$ are essentially distinct points of $\mathbf{P}^2$ and $m_1p_1 + \cdots + m_rp_r$ satisfies the proximity inequalities.

We use divisorial methods on X to obtain results about I_Z. In particular, let f_d denote $de_0 - m_1e_1 - \cdots - m_re_r$ (and let $\mathcal{F}_d$ denote the corresponding line bundle $\mathcal{O}_X(f_d)$). Since $\mathcal{O}_X(e_0)$ is the pullback $\pi^*(\mathcal{O}_{\mathbf{P}^2}(1))$ of the class of a line, using Lemma II.1 of [H] and the projection formula [Ha, Ex. II.5.1(d)], we have for each d and i a natural isomorphism of $H^i(X, \mathcal{F}_d)$ with $H^i(\mathbf{P}^2, \pi_*(\mathcal{O}_X(-m_1e_1-\cdots-m_re_r))\otimes\mathcal{O}(d)) = H^i(\mathbf{P}^2, \mathcal{I}_Z(d))$, where $Z = m_1p_1 + \cdots + m_rp_r$. In particular, the homogeneous coordinate ring $k[\mathbf{P}^2] = \bigoplus_{d\geq 0} H^0(\mathbf{P}^2, \mathcal{O}(d))$ can be identified with $\bigoplus_{d\geq 0} H^0(X, \mathcal{O}_X(de_0))$, and the homogeneous ideal $I_Z = \bigoplus_{d\geq 0} H^0(\mathbf{P}^2, \mathcal{I}_Z(d))$ for $Z = m_1p_1 + \cdots + m_rp_r$ in $k[\mathbf{P}^2]$ can be identified with $\bigoplus_{d\geq 0} H^0(X, \mathcal{F}_d)$.

We now begin with a lemma which we will apply later.

Lemma 1: *Let $p_1, \ldots, p_r$ be essentially distinct points of $\mathbf{P}^2$, X the surface obtained by blowing them up, $e_0, \ldots, e_r$ the corresponding exceptional configuration. Then, for d sufficiently large, $f_d = de_0 - (m_1e_1 + \cdots + m_re_r)$ is the class of an effective divisor moving in a complete linear system which is fixed component free if and only if $-(m_1e_1 + \cdots + m_re_r)$ satisfies the proximity inequalities.*

Proof: By Lemmas II.7 and II.9 of [H], effectivity and being fixed component free follows for $d \geq m_1 + \cdots + m_r$, if $-(m_1e_1 + \cdots + m_re_r)$ satisfies the proximity inequalities. If $-(m_1e_1 + \cdots + m_re_r)$ does not satisfy the proximity inequalities, then there is an irreducible divisor C which is a component of e_i for some $i > 0$ such that $f_d \cdot C = -(m_1e_1 + \cdots + m_re_r) \cdot C < 0$, and hence the complete linear system $|f_d|$ either is empty or has C as a fixed component. $\diamondsuit$

Given an exceptional configuration $e_0, \ldots, e_r$ on some (necessarily smooth projective rational) surface X, consider the cone $\mathcal{S} \subset \mathrm{Cl}(X)$ of all elements f of $\mathrm{Cl}(X)$ meeting every component of each e_i, $i > 0$ nonnegatively and satisfying $f \cdot e_0 = 0$. Thus $\mathcal{S}$ consists precisely of those nonpositive linear combinations $-(m_1e_1 + \cdots + m_re_r)$ satisfying the proximity inequalities.

Certain elements of $\mathcal{S}$ will be of particular interest, corresponding as we will eventually show to simple complete ideals primary with respect to maximal ideals. To define these elements, note that altogether there are r irreducible components Q_j occurring among the exceptional divisors E_i, $1 \leq i \leq r$. Let $[Q_j]$, $1 \leq j \leq r$, be an enumeration of the classes of these irreducible components. Let q_{ij} be the uniquely determined nonnegative integers with $e_i = \sum_j q_{ij}[Q_j]$. For each j, let $f_j = -\sum_i q_{ij}e_i$. Given any essentially distinct points $p_1, \ldots, p_r$, or, alternatively, any exceptional configuration $e_0, \ldots, e_r$, we will refer to $e_0, f_1, \ldots, f_r$ as the *dual* configuration. This terminology is justified by the following long-known result. (For the statement, recall Kronecker's δ_{ij}, which is equal to 1 if $i = j$, and 0 otherwise.)

Proposition 2: *Let $e_0, \ldots, e_r$ be an exceptional configuration on a surface X, let $e_0, f_1, \ldots, f_r$ be the dual configuration and let $[Q_j]$, $1 \leq j \leq r$, be the classes of the irreducible components of the exceptional loci E_i.*

(a) The classes $f_1, \ldots, f_r$ form a basis for the subgroup of Cl(X) generated by $e_1, \ldots, e_r$.

(b) For every $1 \leq i, j \leq r$, we have $f_i \cdot [Q_j] = \delta_{ij}$. (In particular, the classes f_i satisfy the proximity inequalities).

(c) $\mathcal{S}$ is the free commutative monoid on $f_1, \ldots, f_r$ (i.e., every element of the monoid $\mathcal{S}$ is a nonnegative linear combination of the elements $f_1, \ldots, f_r$ in a unique way).

Proof: Since the classes $e_1, \ldots, e_r$ are orthogonal of self-intersection -1 and since $e_0 \cdot [Q_j] = 0$ for every j, we have $[Q_j] = \sum_{i>0} a_{ij}e_i$ for $a_{ij} = -[Q_j] \cdot e_i$. The proof is now a short matrix calculation.

Note that the transpose of the matrix (a_{ij}) is inverse to (q_{ij}): by definition, $e_i = \sum_j q_{ij}[Q_j]$, hence $e_i = \sum_j q_{ij} \sum_l a_{lj}e_l$, so $\sum_j q_{ij}a_{lj} = \delta_{il}$. Thus $f_1, \ldots, f_r$ and $e_1, \ldots, e_r$ are related by an invertible transformation, hence both give bases, as claimed in (a). Moreover, since a matrix commutes with its inverse, we now have $\sum_i q_{ij}a_{il} = \delta_{jl}$ and thus $f_i \cdot [Q_j] = -(\sum_l q_{li}e_l) \cdot (\sum_t a_{tj}e_t) = \sum_l q_{li}a_{lj} = \delta_{ij}$, as claimed in (b).

As for (c), (b) shows that $f_i \in \mathcal{S}$ for each i, while uniqueness follows since $f_1, \ldots, f_r$ are linearly independent in $\mathrm{Cl}(X)$, so we only need to check that every element f of $\mathcal{S}$ is a nonnegative linear combination of the f_i. Now, $\mathcal{S}$ lies in the subgroup of $\mathrm{Cl}(X)$ generated by $e_1, \ldots, e_r$, but $f_1, \ldots, f_r$ is a basis for the same subgroup. Thus for some integers b_i we have $f = \sum_i b_i f_i$, and by (b) intersecting with $[Q_j]$ gives $f \cdot [Q_j] = b_j$, nonnegative since $f \in \mathcal{S}$. ♢

We now recall the notion of a complete ideal [ZS, p. 353]. We restrict our attention to complete ideals in local rings which occur as stalks of the structure sheaf on a smooth projective surface.

Definition 3: : Let $x \in X$ be a closed point of a smooth projective surface X, $\mathcal{O}_{X,x}$ the stalk of the structure sheaf at x. An ideal $I \subset \mathcal{O}_{X,x}$ is called a *valuation ideal* if there is an ideal I' in a valuation ring R of the function field $k(X)$ of X such that R contains $\mathcal{O}_{X,x}$ and $I = I' \cap \mathcal{O}_{X,x}$. An ideal $J \subset \mathcal{O}_{X,x}$ is called *complete* if it is any intersection of valuation ideals.

We will denote the maximal ideal of $\mathcal{O}_{X,x}$ by m_x. The following result, showing that understanding complete ideals which are primary for m_x is the key to understanding them in general, justifies confining our attention to complete ideals which are primary for m_x.

Proposition 4: *Let $I \neq (1)$ be a complete ideal. Then $I = fJ$ for some element $f \in \mathcal{O}_{X,x}$ and some complete ideal $J \subset \mathcal{O}_{X,x}$ primary for m_x.*

Proof: See p. 362, [ZS, Appendix 5]. ♢

The reader may find a few examples helpful. In these examples R denotes the localization at the maximal ideal (x, y) of the polynomial ring $k[x, y]$.

Example 5: Let $f \in (x, y)$ be an irreducible polynomial. Assigning to a polynomial g the largest t such that f^t divides g determines a valuation on $k(x, y)$, and the corresponding valuation ideals in R are the powers of the ideal (f). More generally, let C be any integral curve on a smooth quasi-projective surface X. Then C determines a discrete valuation ν_C on the function field $k(X)$ of X, whose valuation ring is the local ring $\mathcal{O}_{X,C}$ with maximal ideal corresponding to the generic point of C. (Such a valuation is called in older terminology a prime divisor of the first kind.)

Example 6: Let $I = (x, y)R$ be the maximal ideal. Assigning to a nonzero polynomial $f \in k[x, y] \subset R$ the degree of its term of least total degree in x and y determines a valuation on $k(x, y)$, the I-adic valuation. (A valuation of this sort is the simplest example of what is called in older terminology a prime divisor of the second kind.) The resulting valuation ideals are precisely the ideals I^n, $n \geq 0$. Regarding $k[x, y]$ as the coordinate ring of $\mathbf{A}^2$ and $p \in \mathbf{A}^2$ as the zero locus of $I \cap k[x, y]$, it is convenient to also refer to this valuation as the p-adic valuation.

Our goal is to interpret Zariski's results in the following situation. We have a coherent sheaf $\mathcal{I}$ of ideals on a smooth surface X and we assume the stalks $\mathcal{I}_x \subset \mathcal{O}_{X,x}$ are complete ideals, either primary for m_x or $\mathcal{I}_x = (1)$, for every $x \in X$; in this situation we will simply say $\mathcal{I}$ is *complete*. Since $\mathcal{I}$ is coherent, we have for all but finitely many points x that $\mathcal{I}_x = \mathcal{O}_{X,x}$. In particular, $\mathcal{I}$ is the sheaf of ideals of a 0-dimensional (possibly nonreduced) subscheme $Z_{\mathcal{I}} \subset X$.

We now want to show that each complete sheaf $\mathcal{I}$ on $\mathbf{P}^2$ corresponds to a 0-cycle $\sum_i t_i p_i$ for some set of essentially distinct points $p_1, \ldots, p_r$ of $\mathbf{P}^2$, and that, if $\pi : X \to \mathbf{P}^2$ is the blowing up of the points $p_1, \ldots, p_r$ and $e_0, \ldots, e_r$ the associated exceptional configuration, then $\mathcal{I} = \pi_*(\mathcal{O}_X(-t_1e_1 - \cdots - t_re_r))$.

We begin by recalling Zariski's results about factorization of complete ideals.

Theorem 7: *Let $\mathcal{O}_{X,x}$ be the local ring of an algebraic surface X at a smooth (closed) point x and let m_x be the maximal ideal. If I and J are complete ideals in $\mathcal{O}_{X,x}$ primary for m_x, then IJ is a complete ideal primary for m_x.*

Proof: See Theorem 2', p. 385, [ZS, Appendix 5]. ♢

Definition 8: If, as in Theorem 7, I is a complete ideal in $\mathcal{O}_{X,x}$ primary for m_x, we say I is *simple* if it is not a product of two or more complete ideals primary for m_x.

Theorem 9: *Let $\mathcal{O}_{X,x}$ be the local ring of an algebraic surface X at a smooth (closed) point x and let m_x be the maximal ideal. Any complete ideal in $\mathcal{O}_{X,x}$ primary for m_x is in a unique way a product of simple complete ideals primary for m_x.*

Proof: See Theorem 3, p. 386, [ZS, Appendix 5]. ◊

We now consider more examples. As before, R denotes the localization of $k[x,y]$ at (x,y).

Example 10: Zariski [Z, p. 172] shows that while (x,y^2) and (x^2,y) are valuation ideals (see Example V.12 below), $I=(x,y^2)(x^2,y)$ is not. Since products of complete deals are complete, I is a complete ideal which is not a valuation ideal.

Example 11: The ideal $I=(x^2,y^2)$ is not a complete ideal. For suppose I is contained in a valuation ideal J corresponding to some valuation ν of $k(x,y)$ nonnegative on R. Either $\nu(x)\geq\nu(y)$ or $\nu(y)\geq\nu(x)$; say $\nu(x)\geq\nu(y)$. Then $\nu(xy)\geq\nu(y^2)$ so $xy\in J$. Since this holds for any valuation ideal containing I we see that I cannot be a complete ideal (that is, not an intersection of valuation ideals) since xy is not an element of I.

Example 12: Suppose we blow up the point $p=(0,0)\in\mathbf{A}^2$. The points of the exceptional locus of the blowing up correspond bijectively to linear forms $l=ax+by$, with $a,b\in k$. Choose such an l, and let q be the corresponding point of the exceptional locus; thus p,q are essentially distinct points of $\mathbf{A}^2$. If $l\neq x$, then $k[x,l/x]\subset k(x,y)$ is the coordinate ring of an affine neighborhood U of q on the blowing up of $\mathbf{A}^2$ at p. Let R' be the localization of $k[x,l/x]$ at the maximal ideal $(x,l/x)$; then $R\subset R'$. The powers $(x,l/x)^nR'$ of the maximal ideal $(x,l/x)R'$ give valuation ideals in R' with respect to the $(x,l/x)$-adic valuation. (In analogy with the terminology introduced in Example 6, we shall refer to this valuation as the q-adic valuation.) Their restrictions to R give valuation ideals in R; the nontrivial ones are precisely the ideals $I_1=(x,l)=(x,y)$, $I_2=(x^2,l)$, $I_3=I_1I_2$, $I_4=I_2^2$, $I_5=I_1I_2^2$, etc. The ideals I_1 and I_2 are simple complete ideals, and we have the explicit factorizations $I_{2l+1}=I_1I_l^2$ and $I_{2l}=I_l^2$. In fact, for any nonnegative integers a and b, $I_1^aI_2^b$ is also a complete ideal; by Zariski's unique factorization result, different values of a and b always give different ideals. Moreover, the ideal $I_1^aI_2^b$ defines a coherent sheaf $\mathcal{I}$ of ideals on $\mathbf{P}^2$ which on the blowing up X of $\mathbf{P}^2$ at p and q becomes the divisorial sheaf of ideals $\mathcal{O}_X(-(a+b)e_1-be_2)$. (Thus the multiplicative submonoid generated by I_1 and I_2 in the monoid of all complete ideals in R corresponds isomorphically to the additive monoid generated in $\mathrm{Cl}(X)$ by $-e_1$ and $-e_1-e_2$.) The polynomials in $I_1^aI_2^b$ of degree at most n can be naturally identified with $H^0(X,\mathcal{O}_X(ne_0-(a+b)e_1-be_2))$.

Example 13: By iterating the procedure outlined in Example 12, given any set $p_1,\ldots,p_r$ of essentially distinct points we can define the p_r-adic valuation. In particular, let $s=p_1,\ldots,p_r=p$ be essentially distinct points of a smooth surface S with p infinitely near s, let $X=X_{r-1}\to\cdots\to X_0=S$ be the morphisms obtained by blowing up $p_1,\ldots,p_{r-1}$, and, in turn, let $e_1,\ldots,e_{r-1}$ be the corresponding exceptional classes. We have the maximal ideal $m_p\subset\mathcal{O}_{X,p}$ in the stalk of the stucture sheaf of X at p, from which we obtain the m_p-adic valuation ν_p, defined by associating to any $f\in\mathcal{O}_{X,p}$ the least integer t such that $f\in m_p^t-m_p^{t+1}$. We denote the corresponding valuation ring by R_{ν_p}. The ideals $m_p^t\cap\mathcal{O}_{S,s}$ are then valuation ideals (consisting of those elements of $\mathcal{O}_{S,s}$ of m_p-adic value at least t), and in particular are complete ideals.

Example 13 is typical; every simple complete ideal arises in this way:

Theorem 14: *Let $\mathcal{O}_{S,s}$ be the local ring of an algebraic surface at a smooth (closed) point s, and let $I\subset\mathcal{O}_{S,s}$ be a simple complete ideal primary to the maximal ideal m_s. Then there are essentially distinct points $s=p_1,\ldots,p_r=p$ of S such that, in the notation of Example 13, $I=m_p^t\cap\mathcal{O}_{S,s}$ for some $t>0$.*

Proof: By 5(C) of Appendix 5, p. 389, [ZS], I is a valuation ideal for the m_p-adic valuation with respect to some essentially distinct points $p_1,\ldots,p_r=p$ of S. But, in the notation of Example 13, the ideals of the associated valuation ring R_{ν_p} clearly contract in $\mathcal{O}_{X,p}$ to the powers of the maximal ideal m_p, and thus I is an intersection of some power m_p^t with $\mathcal{O}_{S,s}$, where the intersection is defined with respect to the natural inclusion $\mathcal{O}_{S,s} \subset \mathcal{O}_{X,p}$. $\diamondsuit$

We pause for a moment to consider Theorem 14 and its proof in case $S = \mathbf{P}^2$. Let $X = X_r \to \cdots \to X_0 = S$ be the morphisms obtained by blowing up the essentially distinct points $p_1, \ldots, p_r = p$ of the proof of Theorem 14, and, in turn, let $e_0, e_1, \ldots, e_r$ be the corresponding exceptional configuration. The stalk $\mathcal{O}_{X,x}$ at the generic point x of the curve E_r (whose class is e_r) is a discrete valuation ring in $k(X) = k(S)$. It is easy to see that this ring and the associated valuation are exactly the same as in the m_p-adic valuation, and from this perspective the ideal $m_p^t \cap \mathcal{O}_{S,s}$ is just the stalk $\pi_*(\mathcal{O}_X(-te_r))_s$ at s of $\pi_*(\mathcal{O}_X(-te_r))$, regarded as an ideal of $\mathcal{O}_{S,s}$ via $\pi_*(\mathcal{O}_X(-te_r)) \subset \pi_*\mathcal{O}_X = \mathcal{O}_S$.

Suppose p_r is infinitely near to but is not itself a point of S; after reindexing we may assume that p_r is infinitely near to $p_1 \in S$. Then $e_1 - e_r$ is the class of an effective divisor. Since $(e_1 - e_r) \cdot (ne_0 - te_r) < 0$, one of the irreducible components, call it C_1, of the exceptional locus of p_1 is a fixed component of the linear system $|ne_0 - te_r|$, for every $n > 0$ for which it is nonempty. In particular, $\pi_*(\mathcal{O}_X(-te_r))_s = \pi_*(\mathcal{O}_X(-te_r) \otimes \mathcal{O}_X(-C_1))_s$. But again $-te_r - [C_1]$ may meet a component C_2 of e_1 negatively and again we may subtract it off and still obtain the same complete ideal $\pi_*(\mathcal{O}_X(-te_r))_p = \pi_*(\mathcal{O}_X(-te_r) \otimes \mathcal{O}_X(-C_1 - C_2))_p$. Eventually we obtain a class $-t_1e_1 - \cdots - t_re_r$ which meets every component of e_1 nonnegatively (hence meets each e_i nonnegatively, so each t_i is nonnegative) but also satisfies $\pi_*(\mathcal{O}_X(-te_r))_p = \pi_*(\mathcal{O}_X(-t_1e_1 - \cdots - t_re_r))_p$. Thus $-t_1e_1 - \cdots - t_re_r$ is in the dual cone $\mathcal{S}$ generated by the dual configuration $f_1, \ldots, f_r$. In particular, every valuation ideal $m_p^t \cap \mathcal{O}_{S,s}$ is of the form $\pi_*(\mathcal{O}_X(a_1f_1 + \ldots + a_rf_r))$, with respect to some nonnegative integers a_i and some dual configuration coming from some essentially distinct points $p_1, \ldots, p_r$. We now have a converse, and more.

Theorem 15: *Let $s = p_1, \ldots, p_r$ be essentially distinct points of $S = \mathbf{P}^2$, each p_i being infinitely near p_1. Let $e_0, \ldots, e_r$ be the corresponding exceptional configuration on the blowing up $\pi : X \to S$ of the points, and let $f_1, \ldots, f_r$ be the dual classes. Then $\pi_*\mathcal{O}_X(f_i)$ is a simple complete ideal of $\mathcal{O}_{S,s}$ for each i, and π_* induces an injective homomorphism from the additive submonoid of Cl(X) generated by $f_1, \ldots, f_r$ to the multiplicative monoid of complete ideals in $\mathcal{O}_{S,s}$.*

Proof: First we show that the ideals $\pi_*\mathcal{O}_X(f_i)$, $1 \le i \le r$ are complete. Let $[Q_{ji}]$, $1 \le j \le i$, be the class of the proper transform Q_{ji} of $E_j \subset X_j$ on X_i, where $X = X_r \to \cdots \to X_0 = S$ is the factorization of π corresponding to our given exceptional configuration. For each $j < i$, let q_{ji} be the multiplicity of $Q_{ii} = E_i$ as a component of (the total transform of) E_j on X_i. From the remarks preceding Proposition 2 we know $f_i = \sum_{1\le j\le i} -q_{ji}e_j$. Clearly, $\sum_{1\le j<i} q_{ji}(e_j - q_{ji}e_i)$ is a nonegative linear combination of the classes $[Q_{ji}]$, $1 \le j < i$, and we have $-(\sum_{1\le j\le i} q_{ji}^2)e_i = f_i + \sum_{1\le j<i} q_{ji}(e_j - q_{ji}e_i)$. Moreover, since the span in Cl(X) of the classes $[Q_{ji}]$ is negative definite, any nontrivial nonnegative linear combination of them meets one of them negatively. Since $[Q_{ji}]$, $1 \le j < i$, is, by Proposition 2(b), perpendicular to f_i, we see that subtracting off classes $[Q_{ji}]$, $1 \le j < i$, which meet $-(\sum_{1\le j\le i} q_{ji}^2)e_i = f_i + \sum_{1\le j<i} q_{ji}(e_j - q_{ji}e_i)$ negatively leaves f_i. In particular, $(\pi_*\mathcal{O}_X(f_i))_s = (\pi_*(\mathcal{O}_X(-(\sum_{1\le j\le i} q_{ji}^2)e_i)))_s$, and, as we saw in Theorem 14 and the following remarks, the latter is a complete ideal, since it is a valuation ideal for the p_i-adic valuation.

Next we show that π_* induces an injective homomorphism from the additive submonoid of Cl(X) generated by $f_1, \ldots, f_r$ to the multiplicative monoid of complete ideals in $\mathcal{O}_{S,s}$. First we check for any nonnegative integers t_i that $(\pi_*\mathcal{O}_X(f_1))_s^{t_1} \cdots (\pi_*\mathcal{O}_X(f_r))_s^{t_r} = (\pi_*(\mathcal{O}_X(t_1f_1 + \cdots + t_rf_r)))_s$. By Theorem 7, $(\pi_*\mathcal{O}_X(f_1))_s^{t_1} \cdots (\pi_*\mathcal{O}_X(f_r))_s^{t_r}$ is complete; since the containment $\subset$ is clear, it is enough to show that any valuation ideal in $\mathcal{O}_{S,s}$ which contains $(\pi_*\mathcal{O}_X(f_1))_s^{t_1} \cdots (\pi_*\mathcal{O}_X(f_r))_s^{t_r}$ also contains $(\pi_*(\mathcal{O}_X(t_1f_1 + \cdots + t_rf_r)))_s$. In fact, by Theorem 3 of [ZS, Appendix 4] and its proof, it suffices to consider essential valuations, and so, given any p-adic valuation ν_p for a point p infinitely near to s, it suffices to show that $\nu_p((\pi_*(\mathcal{O}_X(t_1f_1 + \cdots + t_rf_r)))_s) = \nu_p((\pi_*\mathcal{O}_X(f_1))_s^{t_1} \cdots (\pi_*\mathcal{O}_X(f_r))_s^{t_r})$.

By blowing up more points if necessary we may assume that p is p_i for some $1 \le i \le r$. Clearly, $\nu_p((\pi_*\mathcal{O}_X(f_1))_s^{t_1} \cdots (\pi_*\mathcal{O}_X(f_r))_s^{t_r}) = \sum_j t_j\nu_p((\pi_*\mathcal{O}_X(f_j))_s)$, and $\nu_p((\pi_*\mathcal{O}_X(f_j))_s) = \sum_{1\le l\le i} q_{li}e_l \cdot f_j$. Likewise, $\nu_p((\pi_*(\mathcal{O}_X(t_1f_1 + \cdots + t_rf_r)))_s) = \sum_{1\le l\le i} q_{li}e_l \cdot (t_1f_1 + \cdots + t_rf_r)$. But the latter simplifies to $\sum_{1\le l\le i}\sum_{1\le j\le r} t_jq_{li}e_l \cdot f_j$, while $\nu_p((\pi_*\mathcal{O}_X(f_1))_s^{t_1} \cdots (\pi_*\mathcal{O}_X(f_r))_s^{t_r}) = \sum_{1\le j\le r}\sum_{1\le l\le i} t_jq_{li}e_l \cdot f_j$, as desired.

We also now see that the homomorphism is injective, since different linear combinations of the f_i never have the same valuations for all valuations. Finally, to see that each $\pi_*\mathcal{O}_X(f_i)$ is a *simple* complete ideal, recall for each i that all p_i-adic valuation ideals are in the image under π_* of the monoid generated by $f_1, \ldots, f_r$. By 5(E) of [ZS, Appendix 5], there is a bijection between the simple complete ideals in $\mathcal{O}_{S,s}$ primary to m_s and the points infinitely near to s, and if p is infinitely near to s, then the simple ideal corresponding to p is a p-adic valuation ideal. Thus the image of the monoid generated by $f_1, \ldots, f_r$ has at least r simple ideals, but clearly only the images of $f_1, \ldots, f_r$ themselves can be simple, and so are in fact precisely the set of simple ideals in the image monoid. ♢

Remark 16: In the course of the proof of Theorem 15 we saw that

$$(\pi_*\mathcal{O}_X(f_i))_s = (\pi_*(\mathcal{O}_X(-(\sum_{1\leq j\leq i} q_{ji}^2)e_i)))_s;$$

i.e., that $(\pi_*\mathcal{O}_X(f_i))_s = m_i^t \cap \mathcal{O}_{S,s}$ for $t = \sum_{1\leq j\leq i} q_{ji}^2$ is the simple ideal corresponding to f_i. More generally, for $i \leq l$, a minor modification shows that $(\pi_*\mathcal{O}_X(f_i))_s = m_l^t \cap \mathcal{O}_{S,s}$ for $t = \sum_{1\leq j\leq i} q_{ji}q_{jl}$. ♢

We now characterize the complete ideals in the local ring of a point in $\mathbf{P}^2$.

Corollary 17: *Let $\mathcal{O}_{S,s}$ be the local ring of $S = \mathbf{P}^2$ at a (closed) point s, and let $I \subset \mathcal{O}_{S,s}$ be a complete ideal primary to the maximal ideal m_s. Then there are essentially distinct points $s = p_1, \ldots, p_r = p$ of S such that, in the notation of Theorem 15, $I = \pi_*(\mathcal{O}_X(\sum_{1\leq i\leq r} a_i f_i))$, for uniquely determined nonnegative integers a_i.*

Proof: By Theorem 9, we know there are simple ideals J_l, $1 \leq l \leq t$, and positive integers b_i such that $I = J_1^{b_1} \cdots J_t^{b_t}$. By Theorem 15, Theorem 14, and the remarks following Theorem 14, for each J_i there are essentially distinct points such that J_i is π_* applied to the line bundle corresponding to some element of the cone generated by the dual configuration corresponding to the points. Taking $p_1, \ldots, p_r$ to be essentially distinct points comprising those required for each J_i, and as usual taking $\{f_i\}$ to be the dual configuration corresponding to the points, we thus have $I = \pi_*\mathcal{O}_X(\sum_{1\leq i\leq r} a_i f_i)$ for appropriate nonnegative integers a_i; uniqueness follows by Theorem 15. ♢

Remark 18: Let $p_1, \ldots, p_r$ be essentially distinct points of $\mathbf{P}^2$. Let $\pi : X \to S$ be the blowing up of the points, and let $e_0, f_1, \ldots, f_r$ be the corresponding dual configuration. Let a_i be nonnegative integers, not all 0; then $a_1 f_1 + \cdots + a_r f_r$ meets every component of every exceptional divisor $E_1, \ldots, E_r$ nonnegatively. By Theorem 15, $\pi_*\mathcal{O}_X(a_1 f_1 + \cdots + a_r f_r)$ is a sheaf of complete ideals, clearly defining a 0-dimensional subscheme $Z \subset S$; by Corollary 17 every sheaf of complete ideals defining a 0-dimensional subscheme is of this form. Rewriting $a_1 f_1 + \cdots + a_r f_r$ in terms of the exceptional classes as $-\alpha_1 e_1 - \cdots - \alpha_r e_r$, we can denote Z as $\alpha_1 p_1 + \cdots + \alpha_r p_r$; i.e., any 0-dimensional subscheme Z defined by a sheaf of complete ideals is uniquely of the form $\alpha_1 p_1 + \cdots + \alpha_r p_r$, where $p_1, \ldots, p_r$ are essentially distinct points of $\mathbf{P}^2$ and $\alpha_1, \ldots, \alpha_r$ are negative integers satisfying the proximity inequalities (i.e., such that $\alpha_1 e_1 + \cdots + \alpha_r e_r$ meets every component of each e_i nonnegatively). This shows that our generalized notion of fat point subscheme coincides with that of 0-dimensional subschemes defined by coherent sheaves of complete ideals.

Remark 19: Let $\mathcal{I}$ be a sheaf of complete ideals on the affine plane $\mathbf{A}^2 \subset \mathbf{P}^2$ defining a 0-dimensional zero-locus Z. Then $\mathcal{I}$ is the sheaf corresponding to some ideal $I \subset k[\mathbf{A}^2]$ and I is complete (i.e., the intersection in $k(\mathbf{A}^2)$ of $k[\mathbf{A}^2]$ with valuation ideals in the local rings of $k[\mathbf{A}^2]$). It follows from Corollary 17, since this is a local question, that $\mathcal{I} = \pi_*\mathcal{O}_X(-a_1 e_1 - \cdots - a_r e_r)$ for some essentially distinct points $p_1, \ldots, p_r$ of $\mathbf{A}^2$, where $\pi : X \to \mathbf{A}^2$ is the blowing up of the points $p_1, \ldots, p_r$ and e_i, $i > 0$, is the class of the total transform on X of p_i. But the polynomials in $k[\mathbf{A}^2]$ of degree at most n correspond bijectively to the homogeneous polynomials $H^0(\mathbf{P}^2, \mathcal{O}(n))$ of degree n, and under this correspondence the polynomials in I of degree at most n carry over to the elements of $H^0(X, \mathcal{O}_X(ne_0 - a_1 e_1 - \cdots - a_r e_r))$ (where e_0 is the pullback of the class of a line in $\mathbf{P}^2$). I.e., the homogenization of the ideal of polynomials vanishing on Z is just $\bigoplus_{n\geq 0} H^0(X, \mathcal{O}_X(ne_0 - a_1 e_1 - \cdots - a_r e_r)) \subset \bigoplus_{n\geq 0} H^0(X, \mathcal{O}_X(ne_0)) = \bigoplus_{n\geq 0} H^0(\mathbf{P}^2, \mathcal{O}(n)) = k[\mathbf{P}^2]$. This reflects Zariski's original formulation [Z, p. 193] of complete ideals in a polynomial ring in 2 indeterminates as "… those and only those ideals whose elements are subject to given base conditions, and to no other conditions. In other words, the polynomials which belong to a complete ideal and whose degree is not greater than a given integer n, form, for any n, a complete linear system."

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