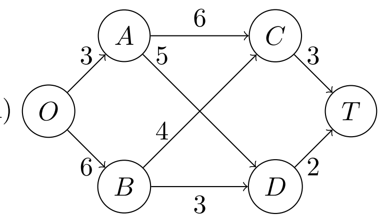
MATH 428/828 Final Exam **Spring Semester, 2013**

Name:____________________ Circle One: **Math428** or **Math828** Score:________

Closed book exam, but calculators are allowed.

1(25pts) Consider the dynamic programming problem for *shortest-path* depicted on the right where the numbers along the edges represent distances between the nodes. Formulate a binary linear programming problem to find the shortest path from the origin to the destination. In particular, show the following



(a) Give precise definition for each binary decision variable. For consistency, use double subscript in x_{ij} so that the first subscript i is for stage i and the second subscript j is for a given link between two nodes. Then define the objective function.

(b) List all exclusionary constraints.

(c) List all contingent constraints. (**Do Not Solve the BIP Problem.**)

2(30pts) Dave's Photography Store has this simple weekly inventory policy for one brand of camera: to have 2 cameras in the beginning of every week. Assume the demand for this particular brand of camera is of Poisson distribution, $P\{D_t = n\} = \frac{\mu^n e^{-\mu}}{n!}$, for a mean of one camera per week, where D_t is the demand random variable for week t.

(a) Let X_t be the number of cameras in stock at the end of week t. Explain why the stochastic process $\{X_t\}$ is a Markov process and find the 1-step transition matrix.

(b) Find the probability that the store has 2 cameras at the end of week 3 given that the store has 1 camera at the end of of week 1.

(c) Find the probability that it takes exactly two weeks from the store's having to order 2 cameras to ordering none.

(d) Find the expected first passage time from the store's having to order 2 cameras to ordering none.

(e) **For Math828 Students:** Let S_t be the number of cameras sold in week t. Find the expected value of S_t, i.e. the expected weekly sales.

3(25pts) The Maytag appliances company offers a complete replacement warranty to a brand of its washing machines if it fails within 2 years. Based on historical data, the company has noted that only 1 percent of the washing machines fail during the first year, whereas 5 percent of the machines that survive the first year will fail during the second year. The warranty covers any replacement as if it is new.

(a) Formulate the evolution of the status of a washing machine as a Markov chain on 4 states: one is to claim the replacement warrant, one is to be in the first year warranty coverage, one is to be in the second year warranty coverage, and the last is to have survived the two-year warranty period. Construct the (one-step) transition matrix. Consider only the transitions of the original warranties, excluding any replacement warranty.

(b) Identify the absorbing state(s).

(c) Find the probability that the company will have to honor the warranty.

(*Continue on Back ...*)

4(25pts) A small supermarket store uses one casher before 5 o'clock in any week day. During this off-peak hours, customers are ready to check out at a rate of 10 customers per hour and the casher takes on average 2 minutes to complete the check-out for each customer. However, in the peak-hour between 5 o'clock and 6 o'clock, customers arrives on average every two minutes but the service rate for each check-out lane remains the same. Assume both the arrival and the check-out processes are exponentially distributed but independent from each other.

(a) Find the check-out waiting time W_q before the peak-hour.

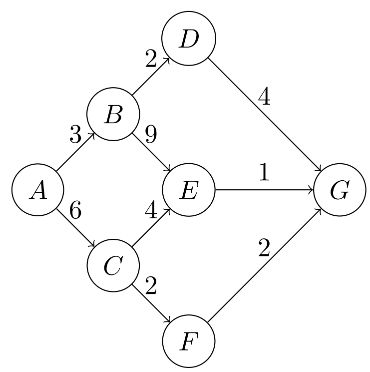
(b) Find the check-out waiting time W_q during the peak-hour if two check-out lanes are open.

(c) **For Math828 Students:** How much time would it save for each customer on average if three check-out lanes instead of two are open during the peak-hour? Round up your answer in seconds.

Useful performance parameters for $M/M/s$ model:

$\rho = \lambda/(s\mu)$	$P_0 = 1\Big/\left[\sum_{n=0}^{s-1}\frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^s}{s!}\frac{1}{1-\rho}\right]$
$L_q = \frac{P_0(\lambda/\mu)^s\rho}{s!(1-\rho)^2}$	$W_q = \frac{L_q}{\lambda}$
$W = W_q + \frac{1}{\mu}$	$L = \lambda\left(W_q + \frac{1}{\mu}\right) = L_q + \frac{\lambda}{\mu}$

5(20pts) Use the backward iteration method for dynamic programming to find the directed path from node A to node G so that the **product** of the edge numbers is the largest. You can use either the graphical or the algebraic method. A correct solution by any other method gets only half of the point.



6(25pts) For a primal LP problem: max. $z = c^T x$ subject to $Ax \le b,\ x \ge 0$, an intermediate step of the simplex method is given as follows:

		z	x_1	x_2	x_3	x_4	x_5	rhs
	z	1	0	-2	5	0	0	20
basic variable	x_1	0	1	0	1	0	0	4
	x_4	0	0	3	-1	1	0	11
	x_5	0	0	1	-2	0	1	2

(a) Complete the iteration to find the optimal solution. Your work must show which is the entering variable, which is the leaving variable, and the reason for each.

(b) From the final tableau found from (a) above, find the solution, $\{w^*, y^*\}$, to the dual LP problem: min. $w = b^T y$ subject to $A^T y \ge c,\ y \ge 0$

(c) **For Math828 Students:** Assume x_3, x_4, x_5 are the slack variables, find the matrix A.

Solution Keys:

1 $x_{ij} = 1$ if path $i - j$ taken, 0 otherwise. Min. $z = 3x_{OA} + 6x_{OB} + 6x_{AC} + 5x_{AD} + 4x_{BC} + 5x_{BD} + 3x_{CT} + 2x_{DT}$ subject to: (exclusionary constraints) $x_{OA} + x_{OB} = 1, x_{AC} + x_{AD} + x_{BC} + x_{BD} = 1, x_{CT} + x_{DT} = 1$, (contingent constraints) $x_{AC} + x_{AD} \le x_{OA}, x_{BC} + x_{BD} \le x_{OB}, x_{CT} \le x_{AC} + x_{BC}, x_{DT} \le x_{AD} + x_{BD}$.

2 (a) $P = \begin{bmatrix} .264 & .368 & .368 \\ .264 & .368 & .368 \\ .264 & .368 & .368 \end{bmatrix}$ (b) $P^2 = P$, so $P\{X_{t+2} = 2|X_t = 1\} = p_{12} = .368$. (c) $P\{T_{02} = 2\} = p_{00}P\{T_{02} = 1\} + p_{01}P\{T_{12} = 1\} = p_{00}p_{02} + p_{01}p_{12} = 0.233$. (d) $\mu_{02} = \mu_{12} = 2.7$ weeks. (e) $E(S_t) = 0.896$ cameras per week.

3 (a) $P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.01 & 0 & 0.99 & 0 \\ 0.05 & 0 & 0 & 0.95 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. (b) State (3). (c) $0.01 + 0.99(0.05) = 0.0545$. Or consider only the original warranty not any replacement warranty, (a) $P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.01 & 0 & 0.99 & 0 \\ 0.05 & 0 & 0 & 0.95 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. (b) State (0) and (3). (c) Solve $Pf = f$ with $f_{00} = 1, f_{30} = 0$ to have $f_{10} = 0.0545$ for any new sales to claim the warranty.

4 (a) $W_q = 1$ min.; (b) $W_q = 40$ sec.; (c) $W_q(s = 2) - W_q(s = 3) = 35$ sec. (Common syntax error: $\sum_n a_n + b = (\sum_n a_n) + b \ne \sum_n (a_n + b)$)

5 Optimal path $A - B - E - G$ with $z^* = 27$.

6 Final tabuleau after only one iteration:

		z	x_1	x_2	x_3	x_4	x_5	rhs
	z	1	0	0	1	0	2	24
basic variable	x_1	0	1	0	1	0	0	4
	x_4	0	0	0	5	1	-3	5
	x_2	0	0	1	-2	0	1	2

for solution $z^* = 24$, $(x_1^*, x_2^*) = (4, 2)$. (b) $w^* = z^* = 24$, $(y_1^*, y_2^*, y_3^*) = (1, 0, 2)$, coefficients of the (x_3, x_4, x_5) for the final tabuleau of (a). (c) Making the coefficient of x_3 a leading 1 by row operation to recover the initial tabuleau:

		z	x_1	x_2	x_3	x_4	x_5	rhs
	z	1	-5	-2	0	0	0	0
basic variable	x_3	0	1	0	1	0	0	4
	x_4	0	1	3	0	1	0	15
	x_5	0	2	1	0	0	1	10

which gives $A = \begin{bmatrix} 1 & 0 \\ 1 & 3 \\ 2 & 1 \end{bmatrix}$ as well as c and b.

End