Proof of Perron-Frobenius Theorem

Proof. Because $P = [p_{ij}]$ (is irreducible and transitive) has non-zero entries, we have

$$\delta = \min_{ij} p_{ij} > 0.$$

Consider the equation of the ijth entry of $P^{t+1} = [p_{ij}^{(t+1)}] = P^t P$,

$$p_{ij}^{(t+1)} = \sum_k p_{ik}^{(t)} p_{kj}.$$

Let

$$0 < m_i^{(t)} := \min_j p_{ij}^{(t)} \leq \max_j p_{ij}^{(t)} := M_i^{(t)} < 1.$$

Then, we have

$$m_i^{(t+1)} = \min_j \sum_k p_{ik}^{(t)} p_{kj} \geq m_i^{(t)} \sum_k p_{kj} = m_i^{(t)}.$$

i.e., the sequence $\{m_i^{(1)}, m_i^{(2)}, \dots\}$ is non-decreasing. Similarly, the upper bound sequence $\{M_i^{(1)}, M_i^{(2)}, \dots\}$ is non-increasing. As a result, both limits $\lim_{t\to\infty} m_i^{(t)} = m_i \leq M_i = \lim_{t\to\infty} M_i^{(t)}$ exist. We now prove they are equal $m_i = M_i$.

To this end, we consider the difference $M_i^{(t+1)} - m_i^{(t+1)}$:

$$\begin{aligned} M_i^{(t+1)} - m_i^{(t+1)} &= \max_j \textstyle\sum_k p_{ik}^{(t)} p_{kj} - \min_\ell \sum_k p_{ik}^{(t)} p_{k\ell} \\ &= \max_{j,\ell} \textstyle\sum_k p_{ik}^{(t)} (p_{kj} - p_{k\ell}) \\ &= \max_{j,\ell} [\textstyle\sum_k^+ p_{ik}^{(t)} (p_{kj} - p_{k\ell}) + \sum_k^- p_{ik}^{(t)} (p_{kj} - p_{k\ell})] \end{aligned}$$

where $\sum_k^+ p_{ik}^{(t)} (p_{kj} - p_{k\ell})$ means the summation of only the positive terms $p_{kj} - p_{k\ell} > 0$ and similarly $\sum_k^- p_{ik}^{(t)} (p_{kj} - p_{k\ell})$ means the summation of only the negative terms $p_{kj} - p_{k\ell} < 0$. Hence,

$$\begin{aligned} M_i^{(t+1)} - m_i^{(t+1)} &\leq \max_{j,\ell} [M_i^{(t)} \textstyle\sum_k^+ (p_{kj} - p_{k\ell}) + m_i^{(t)} \sum_k^- (p_{kj} - p_{k\ell})] \\ &= \max_{j,\ell} [M_i^{(t)} \textstyle\sum_k^+ (p_{kj} - p_{k\ell}) - m_i^{(t)} \sum_k^- (p_{k\ell} - p_{kj})]. \end{aligned} \tag{1}$$

It is critical to notice the following unexpected equality:

$$\begin{aligned} \textstyle\sum_k^- (p_{k\ell} - p_{kj}) &= \textstyle\sum_k^- p_{k\ell} - \sum_k^- p_{kj} \\ &= 1 - \textstyle\sum_k^+ p_{k\ell} - (1 - \sum_k^+ p_{kj}) \\ &= \textstyle\sum_k^+ (p_{kj} - p_{k\ell}). \end{aligned}$$

Hence, the inequality (1) becomes

$$M_i^{(t+1)} - m_i^{(t+1)} \le (M_i^{(t)} - m_i^{(t)}) \max_{j,\ell} \sum_k^{+} (p_{kj} - p_{k\ell}).$$

Let r be the number of terms for which $p_{kj} - p_{k\ell} > 0$, and s be the number of terms for which $p_{kj} - p_{k\ell} < 0$. Then $r + s \le n$, and more importantly

$$\begin{aligned} \textstyle\sum_k^{+} (p_{kj} - p_{k\ell}) \quad &= \textstyle\sum_k^{+} p_{kj} - \sum_k^{+} p_{k\ell} \\ &= 1 - \textstyle\sum_k^{-} p_{kj} - \sum_k^{+} p_{k\ell} \le 1 - s\delta - r\delta \\ &\le 1 - n\delta < 1. \end{aligned}$$

The estimate for the difference $M_i^{(t+1)} - m_i^{(t+1)}$ at last reduces to

$$M_i^{(t+1)} - m_i^{(t+1)} \le (1 - n\delta)(M_i^{(t)} - m_i^{(t)}) \le (1 - n\delta)^t (M_i^{(1)} - m_i^{(1)}) \to 0,$$

as $t \to \infty$, showing $M_i = m_i := w_i$. As a consequence to the inequality $m_i^{(t)} \le p_{ij}^{(t)} \le M_i^{(t)}$, we have $\lim_{t\to\infty} p_{ij}^{(t)} = w_i$ for all j. This proves the theorem. $\square$

References: Bellman(1997); Berman & Plemmons(1994); Frobenius(1908, 1912); Lancaster & Tismenetsky(1985); Marcus & Minc(1984); Perron(1907); Petersen(1983); Seneta(1973).

Ethier and Kurtz, Markov Processes – Characterization and Convergence.