Proof of Perron-Frobenius Theorem

Let $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n$ with y^T denoting exclusively the transpose of vector y. Let $\|x\| = \max_i\{|x_i|\}$ be the norm. Then the induced operator norm for matrix $A = [a_{ij}]$ is $\|A\| = \max_i\{\sum_j |a_{ij}|\}$.

Consider a Markov's chain on n states with transition probabilities $p_{ij} = \Pr(X_{k+1} = i|X_k = j)$, independent of k, and $P = [p_{ij}]$ the transition matrix. Then $\sum_{i=1}^n p_{ij} = 1$ for all j. Let $p_{ij}^{(t)} = \Pr(X_{k+t} = i|X_k = j)$ and $P^{(t)} = [p_{ij}^{(t)}]$ be the t-step transition probability matrix. Then we have $p_{ij}^{(t)} = \sum_\ell p_{i\ell}^{(t-1)} p_{\ell j}$ for all i, j. In matrix, $P^{(t)} = P^{(t-1)}P = \cdots = P^t$ which is the t-step transition matrix. If $q = (q_1, \ldots, q_n)^T$ is a probability distribution for the Markovian states at a given iterate with $q_i \geq 0, \sum q_i = 1$, then Pq is again a probability distribution for the states at the next iterate. A probability distribution w is said to be a steady state distribution if it is invariant under the transition, i.e. $Pw = w$. Such a distribution must be an eigenvector of P and $\lambda = 1$ must be the corresponding eigenvalue. The existence as well as the uniqueness of the steady state distribution is guaranteed for a class of Markovian chains by the following theorem due to Perron and Frobenius.

Theorem 1. *Let $P = [p_{ij}]$ be a probability transition matrix, i.e. $p_{ij} \geq 0$ and $\sum_{i=1}^n p_{ij} = 1$ for every $j = 1, 2, ..., n$. Assume P is irreducible and transitive in the sense that $p_{ij} > 0$ for all i, j. Then 1 is a simple eigenvalue of P and all other eigenvalues λ satisfy $|\lambda| < 1$. Moreover, the unique eigenvector can be chosen to be a probability vector w and it satisfies $\lim_{t\to\infty} P^t = [w, w, \ldots, w]$. Furthermore, for any probability vector q we have $P^t q \to w$ as $t \to \infty$.*

Proof. We first prove a claim that $\lim_{t\to\infty} p_{ij}^{(t)}$ exist for all i, j and the limit is independent of j, $\lim_{t\to\infty} p_{ij}^{(t)} = w_i$.

Because $P = [p_{ij}]$ (is irreducible and transitive) has non-zero entries, we have

$$\delta = \min_{ij} p_{ij} > 0.$$

Consider the equation of the ijth entry of $P^{t+1} = [p_{ij}^{(t+1)}] = P^t P$,

$$p_{ij}^{(t+1)} = \sum_k p_{ik}^{(t)} p_{kj}.$$

Let

$$0 < m_i^{(t)} := \min_j p_{ij}^{(t)} \leq \max_j p_{ij}^{(t)} := M_i^{(t)} < 1.$$

Then, we have

$$m_i^{(t+1)} = \min_j \sum_k p_{ik}^{(t)} p_{kj} \geq m_i^{(t)} \sum_k p_{kj} = m_i^{(t)}.$$

i.e., the sequence $\{m_i^{(1)}, m_i^{(2)}, \dots\}$ is non-decreasing. Similarly, the upper bound sequence $\{M_i^{(1)}, M_i^{(2)}, \dots\}$ is non-increasing. As a result, both limits $\lim_{t\to\infty} m_i^{(t)} = m_i \leq M_i = \lim_{t\to\infty} M_i^{(t)}$ exist. We now prove they are equal $m_i = M_i$.

To this end, we consider the difference $M_i^{(t+1)} - m_i^{(t+1)}$:

$$\begin{aligned} M_i^{(t+1)} - m_i^{(t+1)} &= \max_j \textstyle\sum_k p_{ik}^{(t)} p_{kj} - \min_\ell \sum_k p_{ik}^{(t)} p_{k\ell} \\ &= \max_{j,\ell} \textstyle\sum_k p_{ik}^{(t)} (p_{kj} - p_{k\ell}) \\ &= \max_{j,\ell} [\textstyle\sum_k p_{ik}^{(t)} (p_{kj} - p_{k\ell})^+ + \sum_k p_{ik}^{(t)} (p_{kj} - p_{k\ell})^-] \\ &\leq \max_{j,\ell} [M_i^{(t)} \textstyle\sum_k (p_{kj} - p_{k\ell})^+ + m_i^{(t)} \sum_k (p_{kj} - p_{k\ell})^-] \end{aligned} \tag{1}$$

where $\sum_k p_{ik}^{(t)} (p_{kj} - p_{k\ell})^+$ means the summation of only the positive terms $p_{kj} - p_{k\ell} > 0$ and similarly $\sum_k p_{ik}^{(t)} (p_{kj} - p_{k\ell})^-$ means the summation of only the negative terms $p_{kj} - p_{k\ell} < 0$.

It is critical to notice the following unexpected equality with the notations $\sum_k^- (p_{kj} - p_{k\ell}) := \sum_k (p_{kj} - p_{k\ell})^-$, $\sum_k^+ (p_{kj} - p_{k\ell}) := \sum_k (p_{kj} - p_{k\ell})^+$:

$$\begin{aligned} \textstyle\sum_k (p_{kj} - p_{k\ell})^- &= \textstyle\sum_k^- (p_{kj} - p_{k\ell}) \\ &= \textstyle\sum_k^- p_{kj} - \sum_k^- p_{k\ell} \\ &= 1 - \textstyle\sum_k^+ p_{kj} - (1 - \sum_k^+ p_{k\ell}) \\ &= \textstyle\sum_k^+ (p_{k\ell} - p_{kj}) \\ &= -\textstyle\sum_k (p_{kj} - p_{k\ell})^+. \end{aligned}$$

Hence, the inequality (1) becomes

$$M_i^{(t+1)} - m_i^{(t+1)} \leq (M_i^{(t)} - m_i^{(t)}) \max_{j,\ell} \sum_k (p_{kj} - p_{k\ell})^+.$$

If $\max_{j,\ell} \sum_k (p_{kj} - p_{k\ell})^+ = 0$, it is done that $M_i^{(t)} = m_i^{(t)}$. Otherwise, for the pair j, ℓ that gives the maximum let r be the number of terms in k for which $p_{kj} - p_{k\ell} > 0$, and s be the number of terms for which $p_{kj} - p_{k\ell} < 0$. Then $r \geq 1$, and $\tilde{n} := r + s \geq 1$ as well as $\tilde{n} \leq n$. More importantly

$$\begin{aligned} \textstyle\sum_k (p_{kj} - p_{k\ell})^+ &= \textstyle\sum_k^+ p_{kj} - \sum_k^+ p_{k\ell} \\ &= 1 - \textstyle\sum_k^- p_{kj} - \sum_k^+ p_{k\ell} \\ &\leq 1 - s\delta - r\delta = 1 - \tilde{n}\delta \\ &\leq 1 - \delta < 1. \end{aligned}$$

The estimate for the difference $M_i^{(t+1)} - m_i^{(t+1)}$ at last reduces to

$$M_i^{(t+1)} - m_i^{(t+1)} \leq (1-\delta)(M_i^{(t)} - m_i^{(t)}) \leq (1-\delta)^t (M_i^{(1)} - m_i^{(1)}) \to 0,$$

as $t \to \infty$, showing $M_i = m_i := w_i$. As a consequence to the inequality $m_i^{(t)} \leq p_{ij}^{(t)} \leq M_i^{(t)}$, we have $\lim_{t\to\infty} p_{ij}^{(t)} = w_i$ for all j. In matrix notation, $\lim_{t\to\infty} P^t = [w, w, \ldots, w]$.

Next, we show the $\lambda = 1$ is an eigenvalue with eigenvector w. In fact from the definition of w above $\lim_{t\to\infty} P^t = [w, w, \ldots, w]$ and thus $[w, w, \ldots, w] = \lim_{t\to\infty} P^t = P \lim_{t\to\infty} P^{t-1} = P[w, w, \ldots, w] = [Pw, Pw, \ldots, Pw]$ showing $Pw = w$.

Next, we show the eigenvalue $\lambda = 1$ is simple. Let $x \neq 0$ be an eigenvector. Then $Px = x$. Apply P to the identity repeatedly to have $P^t x = x$. In limit, $\lim_{t\to\infty} P^t x = [w, w, \ldots, w]x = (w_1 \sum x_j, w_2 \sum x_j, \ldots, w_n \sum x_j)^T = (x_1, x_2, \ldots, x_n)^T$. So $x_i = w_i \sum x_j$ for all i. Because $x \neq 0$, we must have $\bar{x} := \sum x_j \neq 0$, and that all x_i have the same sign. In other words, $x = \bar{x}(w_1, \ldots, w_n)^T = \bar{x} w$ for some constant $\bar{x} \neq 0$, showing that the eigenvector of $\lambda = 1$ is unique up to a constant multiple. Finally, for any probability vector q, the result above shows $\lim_{t\to\infty} P^t q = (w_1 \sum q_j, w_2 \sum q_j, \ldots, w_n \sum q_j)^T = w$.

Next, let λ be an eigenvalue of P. Then it is also an eigenvalue for the transpose P^T. Let x be an eigenvector of λ of P^T. Then $P^T x = \lambda x$ and $\|\lambda x\| = |\lambda|\|x\| = \|P^T x\| \leq \|P^T\|\|x\|$. Since $\|P^T\| = 1$ because $\sum_{i=1}^n p_{ij} = 1$ we have $|\lambda| \leq 1$.

Finally, let x be an eigenvector of an eigenvalue λ. Then we have $\lim_{t\to\infty} P^t x = Wx = (\sum x_j)w$ on one hand and $\lim_{t\to\infty} P^t x = \lim_{t\to\infty} \lambda^t x$ on the other hand. So either $|\lambda| < 1$ in which case $\lim_{t\to\infty} \lambda^t x = 0$ and then $\sum x_j = 0$, or $|\lambda| = 1$ in which case $\lambda = e^{i\theta}$ for some θ and the limit $\lim_{t\to\infty} \lambda^t = \lim_{t\to\infty} e^{i\theta t}$ exists since $\lim_{t\to\infty} e^{i\theta t} x = \lim_{t\to\infty} \lambda^t x = \lim_{t\to\infty} P^t x = (\sum x_j)w$. The latter case holds if and only if $\sum x_j \neq 0$ and $\theta = 0$, i.e. $\lambda = 1$. This shows that all eigenvalues that is not $\lambda = 1$ are inside the unit circle. □

References: Bellman(1997); Berman & Plemmons(1994); Frobenius(1908, 1912); Lancaster & Tismenetsky(1985); Marcus & Minc(1984); Perron(1907); Petersen(1983); Seneta(1973).

Ethier and Kurtz, Markov Processes – Characterization and Convergence.