1(8pts) (a) State the Completeness Axiom.
(b) Use the Completeness Axiom only to prove that every bounded sequence \( \{s_n\} \) has a converging subsequence.

2(10pts) Prove that every Cauchy sequence converges. (Caution: Do not use the result to prove itself.)

3(8pts) Let \( f_n(x) = \frac{x^n}{1 + x^n} \) and \( g_n(x) = \frac{x^n}{1 + x^n \ln n} \).
(a) Does \( f_n \) uniformly converge on \( x \in [0, 1] \)? Justify your answer.
(b) Does \( g_n \) uniformly converge on \( x \in [0, 1] \)? Justify your answer.

4(8pts) Prove that every continuous function \( f \) on a bounded closed interval \([a; b]\) obtains its maximum at a point in \([a, b]\).

5(8pts) Show that if \( \sum |a_k| < \infty \), then the radius of convergence for \( \sum a_k x^k \) must not be smaller than 1.

6(8pts) (a) Complete the statement of Weierstrass’s Approximation Theorem: For every function \( f \) on a bounded closed interval \([a, b]\), there is a sequence of ______ that ______ to \( f \) on ______.
(b) State the definition of derivative for a function \( f \) at a point \( a \).
(c) State the Mean Value Theorem.

7(8pts) Prove the Chain Rule: If \( f \) is differentiable at \( a \) and \( g \) is differentiable at \( f(a) \), then the composite function \( g \circ f \) is differentiable at \( a \) and \( (g \circ f)'(a) = g'(f(a))f'(a) \).

8(8pts) Prove that if \( x_0 \) is a local extremum of a function \( f \) over an open interval containing \( x_0 \) and \( f \) is differentiable at \( x_0 \), then \( f'(x_0) = 0 \).

9(8pts) (a) Show that if \( f'(x) < 0 \) for all \( x \in (a, b) \), then \( f \) is strictly decreasing.
(b) Show that \( x < \tan x \) for all \( x \in (0, \pi/2) \).

10(8pts) Let \( f \) be defined on \( \mathbb{R} \) and suppose there is an \( \epsilon > 0 \) such that
\[
|f(x) - f(y)| \leq |x - y|^{1+\epsilon}.
\]
Prove that \( f \) is a constant function.

11(8pts) Let \( f(x) = \ln(1 + x), |x| < 1 \).
(a) Show that \( f^{(n)}(x) = (-1)^{n+1}(n-1)!(1 + x)^{-n} \).
(b) Use Taylor’s Theorem (see Problem 12 below) only to show
\[
\ln(1 + x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, \quad |x| < 1.
\]

12(10pts) Do (a) or (b) but not both.
(a) State and PROVE Rolle’s Theorem.
(b) Prove Taylor’s Theorem: If \( f^{(n)} \) exists on \((a, b)\) with \( a < 0 < b \) for some \( n \geq 1 \), then for each \( x \in (a, b) \) there is some \( y \) between 0 and \( x \) such that
\[
R_n(x) = \frac{f^{(n)}(y)}{n!} x^n
\]
where \( R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k \).

The End