Note: The homework you turn in must contain each problem statement in its entirety and followed by its solution, as demonstrated in the first problem below. Others below are sketches, outlines, or hints for solutions.

[#1.3] Prove $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$ for all natural numbers n.

Proof: Notice first that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$. So we only need to show $1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$. By induction, we have $1^3 = \frac{1^2(1+1)^2}{4}$ for $n = 1$. So the identity holds for $n = 1$. Assume it holds for n. Now consider the case for $n + 1$. By the induction assumption, we have $1^3 + 2^3 + \cdots + n^3 + (n + 1)^3 = \frac{n^2(n+1)^2}{4} + (n + 1)^3 = (n + 1)^2[\frac{n^2}{4} + (n + 1)] = (n + 1)^2 \frac{n^2+4n+4}{4} = \frac{(n+1)^2(n+2)^2}{4}$, which is the case for $n + 1$. This completes the proof. □

[#1.4 (a) Let $S_n = 1 + 3 + 5 + \cdots + (2n - 1)$. Then $S_1 = 1, S_2 = 4, S_3 = 9$, suggesting $S_n = n^2$. (b) Assume $S_n = n^2$. Then $S_{(n+1)} = S_n + 2(n + 1) - 1 = n^2 + 2n + 1$ by the assumption and simplification, which equals $(n + 1)^2$ by complete squaring. Hence, by induction we have shown that $S_n = n^2$ for all $n \in \mathbb{N}$.

[#1.6 *] Modelled after Example 2 page 3.

[#1.12*] (a) Notice first that for all $n \in \mathbb{N}$, $\binom{n}{0} = \frac{n!}{0!(n-0)!} = 1, \binom{n}{n} = \frac{n!}{n!(n-n)!} = 1$, $n! = n(n-1)\cdots 2\cdot 1 = n(n-1)! = n(n-1)(n-2)!$, $\binom{n}{1} = \frac{n!}{1!(n-1)!} = n, \binom{n}{2} = \frac{n!}{2!(n-2)!} = \frac{n(n-1)}{2}$. With those identities, you can check the cases for $n = 1, 2, 3$ directly. Also, notice that there are $n + 1$ terms in the binomial expansion $(a + b)^n$. (b) $\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-(k-1))!}$ Because $k! = k(k-1)!, (n - (k - 1))! = (n + 1 - k)! = (n + 1 - k)(n - k)!$, the common denominator is $k(k - 1)!(n + 1 - k)(n - k)! = k!(n + 1 - k)!$. Simplify the addition then as follows: $\binom{n}{k} + \binom{n}{k-1} = \frac{n!(n+1-k)+n!k}{k!(n+1-k)!} = \frac{n![(n+1-k)+k]}{k!(n+1-k)!} = \frac{n!(n+1)}{k!(n+1-k)!} = \frac{(n+1)!}{k!(n+1-k)!} = \binom{n+1}{k}$. (c) Use induction. More precisely, the expansion for $n = 1$ trivially. Assume the expansion for n. Then consider the case for $n + 1$: $(a + b)^{n+1} = (a + b)^n(a + b) = \left[\binom{n}{0} a^n + \binom{n}{1} a^{n-1}b + \cdots \binom{n}{n} b^n\right](a + b)$, by the assumption. Expand further, collect like terms $a^{n+1}, a^nb, \ldots, a^kb^{n-k}, b^{n+1}$. You will find with exceptions for the 1st and last terms, the coefficient for $a^kb^{(n-k)}$ is $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ by (b) for $k = 1, 2, \ldots, n$. For a^{n+1}, b^{n+1}, their coefficients remain to be $\binom{n}{0} = \frac{n!}{0!(n-0)!} = \binom{n+1}{0} = 1, \binom{n}{n} = \frac{n!}{0!(n-0)!} = \binom{n+1}{n+1} = 1$.

[#2.2, 2.4, 2.5] They are all similar. Follow Examples 2–6 of §2.

[#3.3]

$$\begin{aligned}(-a)(-b) &= (-a)(-b) + 0 \text{ (by A3)}\\ &= (-a)(-b) + (ab + (-ab)) \text{ (A4)}\\ &= [(-a)(-b) + (-ab)] + ab \text{ (A2, A1)}\\ &= [(-a)(-b) + (-a)b)] + ab \text{ (Thm 3.1(iii))}\\ &= (-a)[(-b) + b)] + ab \text{ (DL)}\\ &= (-a)0 + ab = ab \text{ (A4, A3)}\end{aligned}$$

[#3.5] (a) First it is obvious that $-|b| \le b \le |b|$ because either $|b| = b$ or $|b| = -b$ which implies for the former case that $|b| = b \ge 0 \ge -|b|$ and that $-|b| = b \le 0 \le |b|$ for the latter case. Therefore, together with $|b| \le a$ it implies $-a \le -|b| \le b \le |b| \le a$ as required. Conversely, if $-a \le b \le a$, then we must have $|b| = b \le a$ if $b \ge 0$ using the right part of the inequality $b \le a$, or $|b| = -b \le a$ if $b \le 0$ using the left part of the inequality $-a \le b \Rightarrow -b \le a$. (b) By (a) we only need to show $-|a-b| \le |a|-|b| \le |a-b|$. For the right part, we have $|a| = |a-b+b| \le |a-b|+|b|$ by the triangle inequality. Hence, $|a| - |b| \le |a - b|$. This is true for all a, b. Exchanging a, b, we have $|b| - |a| \le |b - a| = | - (b - a)| = |a - b|$ which is the same as $-|a - b| \le |a| - |b|$, showing the left part of the inequality.

[#3.6] (a) $|a + b + c| = |a + (b + c)| \le |a| + |b + c| \le |a| + |b| + |c|$, using the triangle inequality twice in a roll. (b) Follow the hint.

[#3.7*] (a) Same as 3.5(a) changing $\leq$ to $<$ in the argument. Alternatively, it is a special case of 3.5(a). More precisely, for $|b| < a$, we cannot have $b = a$ if $b \geq 0$ nor $b = -a$ since $-a < -|b| = b$. Thus, $|b| < a$ iff $-a \leq b \leq a$ with $b \neq a, -a$ iff $-a < b < a$. (b) By (a), $|a - b| < c \iff -c < a - b < c \iff b - c < a < b + c$. (c) Same as (b), changing $<$ to $\leq$ in the argument. Alternative, consider the two cases $|a - b| < c$ and $|a - b| = c$ separately. For the former, $|a - b| < c \Longrightarrow b - c < a < b + c$ by (b), $\Longrightarrow b - c \leq a \leq b + c$. Conversely, $|a - b| < c$ implies $a - b \neq c, -c$, which together with $b - c \leq a \leq b + c$ implies $b - c < a < b + c \Longrightarrow |a - b| < c$ by (b). For the latter case that $|a - b| = c$, either $a - b = c$ or $-c$, implying $a = b + c$ or $b - c$, implying $b - c \leq a \leq b + c$. Conversely, $b - c \leq a \leq b + c \Longrightarrow -c \leq a - b \leq c$. Together with the case definition $|a - b| = c$ we have the trivial conclusion $|a - b| \leq c$.

[#3.8*] Assume instead that $a > b$. Then $b < 2^{-1}(a + b) < a$ because $2b = b(1 + 1) = b + b < a + b < a + a = 2a$. Let $b_1 = 2^{-1}(a + b) > b$. Then by the hypothesis we have $a \leq b_1 = 2^{-1}(a + b) \to 2a = 2 \cdot 2^{-1}(a + b) = a + b \to a \leq b$, contradicting the assumption that $a > b$.

[#4.5] Since $s \leq m = \sup S$ for all $s \in S$ and $m \in S$, therefore by definition $m = \max S$.

[#4.6] (a) Since $S \neq \emptyset$, $\exists s_0 \in S$ s.t. $\inf S \leq s_0 \leq \sup S$. (b) S must be a one-point set $S = \{a\}$ for some $a \in \mathbb{R}$.

[#4.7*] (a) $\forall s \in S \subset T$, $\inf T \leq s$ by a part of the definition of $\inf T$. This implies $\inf T$ is a lower bound of S. By $\inf S$, we must have $\inf T \leq \inf S$. You then show similarly that $\sup S \leq \sup T$. The part $\inf S \leq \sup S$ is from #4.6. (b) Since $S, T \subset S \cup T$, by (a), $\sup S, \sup T \leq \sup(S \cup T)$ and $\max\{\sup S, \sup T\} \leq \sup(S \cup T)$. One the other hand, $\forall a \in S \sup T$, either $a \in S$ or $a \in T$, which implies either $a \leq \sup S$ or $a \leq \sup T \iff a \leq \max\{\sup S, \sup T\}$. Therefore, $\max\{\sup S, \sup T\}$ is an upper bound of $S \cup T$. Because $\sup(S \cup T)$ is the least upper bound, $\sup(S \cup T) \leq \max\{\sup S, \sup T\}$. Together the established inequality $\max\{\sup S, \sup T\} \leq \sup(S \cup T)$ we have the equality $\max\{\sup S, \sup T\} = \sup(S \cup T)$.

[#4.10*] By Archimedean Property, $\exists k \in \mathbb{N}$ s.t. $ka > 1$ since $a > 0, 1 > 0$. Thus $a > \frac{1}{k}$ since $k > 0$. Use the property for same pair $1, a$, $\exists m \in \mathbb{N} \Rightarrow m = m \cdot 1 > a$. Let $n = \max k, m$, then $\frac{1}{n} \leq \frac{1}{k} < a < m \leq n$.

[#4.11] It suffices to show there is an infinite sequence $a < a_1 < a_2 < \cdots < a_n \cdots < b$ with $a_n \in \mathbb{Q}$. Construct the sequence by induction. By the denseness of $\mathbb{Q}$, $\exists a_1 \in \mathbb{Q} \Rightarrow a < a_1 < b$. Assuming a_n is constructed such that $a < a_1 < a_2 < \ldots a_n <$, then applying the same denseness property to the pair $a_n < b$ to have some $a_{n+1} \in \mathbb{Q}$ with $a_n < a_{n+1} < b$. This completes the proof.

[#4.12] By the denseness property of $\mathbb{Q}$ in $\mathbb{R}$, we have for this pair $a - \sqrt{2} < b - \sqrt{2}$ a rational $r \in \mathbb{Q}$ such that $a - \sqrt{2} < r < b - \sqrt{2}$, which is $a < r + \sqrt{2} < b$. Since $r \in \mathbb{Q}, \sqrt{2} \in \mathbb{I}$, we must $x = r + \sqrt{2}\mathbb{I}$ for otherwise $\sqrt{2} = x - r \in \mathbb{Q}$ would be a contradiction.

[#4.16*] By the way set $A := \{r \in \mathbb{Q} : r < a\}$ is defined, we concluded right away that a is an upper bound of A: $\sup A \leq a$. If $a \neq \sup A$, then we must have $\sup A < a$. By the denseness property of $\mathbb{Q}$, there is a $r \in \mathbb{Q}$ such that $\sup A < r < a$. By definition of A, this $r \in A$ contradicting the implication that $a \leq \sup A$.

[#5.2]

[#5.4*] Consider 2 cases separately. Case of $m = \inf S > -\infty$. Then $\forall s \in S$, $m \leq s$ which implies $-s \leq -m$. Thus $-m$ is an upper bound for $-S$. Moreover, if A is an upper bound of $-S$: $-s \leq A \forall s \in S$, then $-A \leq s \forall s \in S$ is an lower bound of S, and $-A \leq m = \inf S$ follows. Thus $-m \leq A$, and $-m = \sup(-S)$ by definition and $\inf S = m = -(-m) = -(\sup(-S))$ follows. For the remaining case that $\inf S = -\infty$ that S is not bounded below, then $-S$ cannot be bounded above because M is an upper bound of $-S$ iff $-M$ is a lower bound of S. Hence $\sup(-S) = \infty$, and $\inf S = -\infty = -\sup(-S)$.

[#5.5] The argument is identical to 4.6(a).

[#7.2]

[#7.4*] (a) $\sqrt{2}/n \to 0$. (b) $(1 + 1/n)^n \to e$. $t_{n+1} = (t_n^2 + 2)/(2t_n), t_1 = 1$. Then $1 \leq t_n \leq 2$, and t_n increasing. $\lim t_n = t$ exists, and $t = \sqrt{2}$.

[#8.2a,c] (a) $\forall \epsilon > 0$ let $N = 1/\epsilon$. Then $n > N \Longrightarrow |a_n - 0| = a_n = n/(n^2 + 1) < n/n^2 = 1/n < 1/N = \epsilon$.

[#8.4] By assumption, $\forall \epsilon > 0, \exists N > 0$ *s.t.* $n > N \Longrightarrow |s_n| < \epsilon/M, \Longrightarrow |s_n t_n| = |s_n||t_n| < (\epsilon/M)M = \epsilon$ since $|t_n| < M$ for all n.

[#8.8(a)*] (This is another example for how YOUR hand-in homework should look like for this problem: State the problem, followed by a formal declaration "Proof" or "Solution" whichever applies.)
Prove the limit $\lim(\sqrt{n^2+1}-n)=0$.
Proof: $\forall\epsilon>0$, let $N=1/\epsilon$. Then for $n>N$ we have $|\sqrt{n^2+1}-n-0|=\sqrt{n^2+1}-n=(\sqrt{n^2+1}-n)\cdot(\sqrt{n^2+1}+n)/(\sqrt{n^2+1}+n)=(\sqrt{n^2+1}^2-n^2)/(\sqrt{n^2+1}+n)=1/(\sqrt{n^2+1}+n)<1/n<1/N=\epsilon$. This proves $\lim(\sqrt{n^2+1}-n)=0$ by definition. □

[#9.2b] By Theorems 9.2 and 9.3, $\lim(3y_n-x_n)=\lim(3y_n+(-1)x_n)=3\cdot 7+(-1)\cdot 3=18$. By Theorem 9.6, $\lim(3y_n-x_n/y_n=18/7$.

[#9.4] (b) $s_1=1, s_{n+1}=\sqrt{s_n+1}$. Assume $\lim s_n=s$ exists. Than $\lim s_{n+1}=\lim s_n=s$. By Example 5 of § and Theorem 9.3, $s=\lim s_{n+1}=\lim\sqrt{s_n+1}=\sqrt{\lim s_n+1}=\sqrt{s+1}$. Solving s we get $s=(1+\sqrt{5})/2$.

[#9.6a,b] (a) Plug in the "limit" to get $a=3a^2\implies a=0$ or $a=1/3$. (b) The limit does not exits because $x_n>3^{n-1}\to\infty$ by induction.

[#9.8]

[#9.10*] (a) By assumption, $\forall M>0,\exists N,\ s.t.\ n>N\implies s_n>M/k>0$ since $k>0$ is a constant. Therefore $ks_n>k(M/k)=M$ for all $n>N$, showing $ks_n\to\infty$ by definition. (b) ($\Longrightarrow$) $\forall M<0$, $\exists N,\ s.t.\ n>N\implies s_n>-M$ since $\lim s_n=+\infty$. Hence we have $-s_n<M$ showing $\lim(-s)=-\infty$ by definition. Similar argument applies to ($\Longleftarrow$). Also for (c).

[#9.12*] (a) Let $\epsilon_0=(1-L)/2>0$ as $L<1$. By assumption $\exists N_0>0$ such that $\forall n\ge N_0, ||s_{n+1}|/|s_n|-L|<\epsilon_0\iff L-\epsilon_0<|s_{n+1}|/|s_n|<L+\epsilon_0=(1+L)/2$. Let $a=(1+L)/2$. Then $L<a<1$, and $|s_{n+1}|/|s_n|<a$ for $n\ge N_0$. Repeatedly using this inequality for $n,n-1,\ldots,n-(n-N_0)=N_0>\ge N_0$, we have $|s_n|<a|s_{n-1}|<a(a|s_{n-2}|)<\cdots<a^{n-N_0}|s_{n-(n-N_0)}|=a^{n-N_0}|s_{N_0}|$. Because N_0 is fixed and $a^n\to 0$ as $n\to\infty$ since $0<a<1$, we have $\forall\epsilon>0,\exists N$ s.t. $n>N\implies a^n<\epsilon a^{N_0}/|s_{N_0}|\implies|s_n|<a^{n-N_0}|s_{N_0}|<\epsilon$. (b) Let $t_n=1/|s_n|$. Then $t_{n+1}/t_n=1/(s_{n+1}/s_n)\to 1/L<1$. By (a) $\lim t_n=0$ which is equivalent to $s_n\to\infty$ by Theorem 9.10.

[#9.14] Follow the hints.

[#9.16] Follow the instruction.

[#10.6*] (a) $\forall\epsilon>0$, let $N=-\frac{\ln\epsilon}{\ln 2}$, then $n\ge m>N$ implies

$$\begin{aligned}|s_n-s_m|&=|s_n-s_{n-1}+s_{n-1}-s_{n-2}+\cdots-s_{m+1}+s_{m+1}-s_m|\\&\le|s_n-s_{n-1}|+|s_{n-1}-s_{n-2}|+\cdots+|s_{m+1}-s_m|\\&<2^{-(n-1)}+2^{-(n-2)}+\cdots+2^{-(m+1)}+2^{-m}\\&=2^{-m}\frac{1-2^{-(n-m)}}{1-2^{-1}}\\&<2^{-(m-1)}\le 2^{-N}=\epsilon.\end{aligned}$$

(b) No. Conterexample: $s_n=\sum_{k=1}^n 1/k\to\infty$ as $n\to\infty$ hence, it cannot be Cauchy for every Cauchy sequence must be bounded. However $s_{n+1}-s_n=1/(n+1)<1/n$ satisfied.

[#10.10*] (a, b) are straightforward. (c) Assume (s_n) is not nonincreasing, then there is an n such that $s_{n+1}>s_n\iff(s_n+1)/3>s_n\iff s_n<1/2$ contradicting $s_n\ge 1/2$ for all n. (d) Since (s_n) nonincreasing and bounded below by $1/2$, $\lim s_n=s_0\in\mathbb{R}$ exists. Using a limit theorem on $s_{n+1}=(s_n+1)/3$ we have $s_0=\lim s_{n+1}=\lim(s_n+1)/3=(\lim s_n+1)/3=(s_0+1)/3\iff s_0=1/2$.

[#11.8*] (a) Let $S_N=\{s_n:n>N\}$. Then by Ex.5.4, $\inf\{s_n:n>N\}=-\sup(-\{s_n:n>N\})=-\sup\{-s_n:n>N\}$. Taking limit in $N\to\infty$, then by definition and a limit theorem we have $\liminf s_n=\lim_{N\to\infty}\inf\{s_n:n>N\}=\lim_{N\to\infty}(-\sup\{-s_n:n>N\})=-\lim_{N\to\infty}\sup\{-s_n:n>N\}=-\limsup(-s_n)$. (b) Obviously $(-t_k)$ is monotone iff (t_k) is monotone. Then by a limit theorem and (a) we have $\lim_{k\to\infty}(-t_k)=-\lim_{k\to\infty}t_k=-(\limsup(-s_n))=\liminf s_n$.

[#11.10*] (a) $S=\{1/n:n\in\mathbb{N}\}\cup\{0\}$. In fact, the nth column subsequence converges to $1/n$, and every row subsequence converges to 0. So $S\supset\{1/n:n\in\mathbb{N}\}\cup\{0\}$. Moreover, for any number $a\notin S$, there is a small $\epsilon_0>0$ such that the interval $(a-\epsilon_0,a+\epsilon_0)$ contains no points of the sequence (s_n), and therefore a cannot be a subsequential limit of (s_n), and $S=\{1/n:n\in\mathbb{N}\}\cup\{0\}$ follows. (b) By inspection, $\limsup s_n=1=\sup S,\liminf s_n=0=\inf S$.

[#12.2] Since $0\le\liminf|s_n|\le\limsup|s_n|$ for any sequence, then $\limsup|s_n|=0$ iff $\liminf|s_n|=$

$\limsup |s_n| = 0$ iff $\lim |s_n| = 0$ iff $\lim s_n = 0$.
[#12.4] Follow the hint.
[#12.6] Follow the hint.
[#12.8*] Because every sequence has a subsequence converging to its limsup (Rmk: state known results rather than theorem, corollary, or lemma numbers from text, such as Corollary 11.4 in this case. Follow this convention when you take exams), there is a subsequence $r_{n_k} = s_{n_k} t_{n_k}$ of the product sequence $r_n := s_n t_n$ such that $\lim_{k\to\infty} s_{n_k} t_{n_k} = \limsup s_n t_n$. (This does not imply s_{n_k} or t_{n_k} convergent!). Because every bounded sequence has a converging subsequence, and $(s_n), (t_n)$, hence $(s_{n_k}), (t_{n_k})$ automatically, are bounded, (s_{n_k}) has a converging subsequence $(s_{n_{k_l}})$ to $s \in \mathbb{R}$ (This does not imply $(t_{n_{k_l}})$ converges). By the same result, $(t_{n_{k_l}})$ has a converging subsequence $(t_{n_{k_{l_m}}})$ to $t \in \mathbb{R}$. Now we have found convergent subsequences $(t_{n_{k_{l_m}}})$, $(s_{n_{k_{l_m}}})$. Because all subsequences of a convergent sequence converge to the same limit, we have

$$\lim_{m\to\infty} s_{n_{k_{l_m}}} t_{n_{k_{l_m}}} = \lim_{k\to\infty} s_{n_k} t_{n_k} = \limsup s_n t_n.$$

On the other hand, by the limit product theorem we have

$$\begin{aligned}\lim_{m\to\infty} s_{n_{k_{l_m}}} t_{n_{k_{l_m}}} &= \lim_{m\to\infty} s_{n_{k_{l_m}}} \lim_{m\to\infty} t_{n_{k_{l_m}}} \\ &\le \limsup s_n \limsup t_n.\end{aligned}$$

The last inequality holds because lim sup of every sequence is the least upper bound of all the sequential limits of the sequence, and the fact that both s_n, t_n are nonnegative.
A simpler, alternative proof by Kirsty: For any $n > N$ we have $S_N = \sup\{s_n : n > N\} \ge s_n \ge 0, T_N = \sup\{t_n : n > N\} \ge t_n \ge 0 \Longrightarrow s_n t_n \le S_N T_N \Longrightarrow \sup\{s_n t_n : n > N\} \le S_N T_N$. By definition of lim sup and the product limit theorem, we have

$$\begin{aligned}\limsup s_n t_n &= \lim_{N\to\infty} \{s_n t_n : n > N\} \\ &\le \lim_{N\to\infty} S_N T_N = \lim_{N\to\infty} S_N \lim_{N\to\infty} T_N \\ &= \limsup s_n \limsup t_n. \qquad \square\end{aligned}$$

[#12.12*] Following the hint, we have for any $n > M > N$,

$$\begin{aligned}\sigma_n &= \frac{s_1 + \cdots + s_n}{n} = \frac{s_1 + \cdots + s_N}{n} + \frac{s_{N+1} + \cdots + s_n}{n} \\ &\le \frac{s_1 + \cdots + s_N}{M} + \frac{n-N}{n} \sup\{s_n : n > N\} \\ &\text{(for } s_n \ge 0, n > M) \\ &< \frac{s_1 + \cdots + s_N}{M} + \sup\{s_n : n > N\} \\ &\text{(for } \tfrac{n-N}{n} < 1, n > N \text{ and } s_n \ge 0)\end{aligned} \tag{1}$$

Since it holds for all $n > M$, it holds for $\sup\{\sigma_n : n > M\}$

$$\sup\{\sigma_n : n > M\} \le \frac{s_1 + \cdots s_N}{M} + \sup\{s_n : n > N\}$$

By definition and the fact that limits preserve inequality relations we have $\limsup \sigma_n = \lim_{M\to\infty} \sup\{\sigma_n : n > M\} \le \lim_{M\to\infty}[\frac{s_1+\cdots s_N}{M} + \sup\{s_n : n > N\}] = \sup\{s_n : n > N\}$. Since this inequality holds for all N,$\limsup \sigma_n \le \lim_{N\to\infty} \sup\{s_n : n > N\} = \limsup s_n$ follows. To show $\liminf s_n \le \liminf \sigma_n$, we argue similarly as in (**??**) above as follows:

$$\begin{aligned}\sigma_n &= \frac{s_1 + \cdots s_n}{n} = \frac{s_1 + \cdots + s_N}{n} + \frac{s_{N+1} + \cdots + s_n}{n} \\ &\ge \frac{n-N}{n} \inf\{s_n : n > N\} \text{(for } s_n \ge 0) \\ &= 1 - \frac{N}{n} \inf\{s_n : n > N\} \\ &> 1 - \frac{N}{M} \inf\{s_n : n > N\} \text{ (for } s_n \ge 0, n > M)\end{aligned}$$

Taking the limits in the order of $M \to \infty$ first and $N \to \infty$ afterwards gives rise to the required result.
[#14.4] (a) Use Comparison Test. $1/[n + (-1)^n]^2 \le 1/(n-1)^2$ and $\sum_{n=2}^\infty 1/(n-1)^2 = \sum_{n=1}^\infty 1/n^2$ converges. (b) The partial sum $s_n = (\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) + \cdots + (\sqrt{n} - \sqrt{n-1}) + (\sqrt{n+1} - \sqrt{n}) = \sqrt{n+1} - 1 \to \infty$. Diverges. (c) By Ratio Test, $|a_{n+1}/a_n| = 1/(1 + 1/n)^n \to 1/e < 1 \Longrightarrow$ converges.
[#14.6] Let B be an upper bound of $(|b_n|)$: $|b_n| \le B \forall n$. By Cauchy criterion and the assumption that $\sum |a_n| < \infty$, we have $\forall \epsilon > 0, \exists N$ s.t. $\forall m > n > N \Longrightarrow \sum_{k=n+1}^m |a_k| < \epsilon/B$. Hence $\sum_{k=n+1}^m |a_k b_k| \le$

$B\sum_{k=n+1}^{m}|a_k| < B\epsilon/B = \epsilon$. This proves by Cauchy criterion that $\sum a_n b_n$ converges absolutely.

[#14.8] Use this inequality: $(a+b)^2 = a^2+2ab+b^2 \geq ab$ and the Comparison Test.

[#14.12*] Since every sequence has a converging subsequence to its liminf, we have in this case a subsequence a_{n_k} such that $|a_{n_k}| \to \liminf |a_n| = 0$. Thus, w.l.o.g., we assume $|a_n| \to 0$ as $n \to \infty$. We next construct a subsequence a_{n_k} such that $|a_{n_k}| \leq \frac{1}{k^2}$ (without specifying, it implies automatically that $n_1 < n_2 < \cdots < n_k < \cdots$.) We do this by induction using the assumption that $a_n \to 0$. By definition, for $\epsilon = 1/1^2 = 1$, $\exists N$ s.t. $n > N \implies |a_n| = |a_n - 0| < \epsilon = 1/1^2$. Define $n_1 = N+1$. Assume $a_{n_i}, i = 1, 2, \ldots, k$ are found. Then to construct $a_{n_{k+1}}$ we use again the assumption that $a_n \to 0$. To this end, let $\epsilon = 1/(k+1)^2$. Then $\exists N$ s.t. $n > N \implies |a_n| < \epsilon = 1/(k+1)^2$. Define $n_{k+1} = \max\{N+1, n_k+1\}$ then we have $n_{k+1} > n_k$ and $|a_{n_{k+1}}| < 1/(k+1)^2$ as required. Hence by induction (a_{n_k}) can be constructed with $|a_{n_k}| < 1/k^2$. Since $\sum \frac{1}{k^2} < \infty$ converges, by the Comparison Test, $\sum_{k=1}^{\infty} a_{n_k}$ converges absolutely, and itself converges as well.

[#14.14*] Let s_n be the nth partial sum of this series $\sum a_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \cdots$. Then s_n is a monotone increasing sequence. Notice that there are exactly 2^{k-1} terms of the form $\frac{1}{2^k}$, all together there are $1 + 2 + 2^2 + \cdots 2^{k-1} = 2^k - 1$ terms for all the terms having the form $\frac{1}{2^i}$ with $i = 1, 2, \ldots, k$. Hence the (2^k-1)st partial sum is

$$s_{2^k-1} = \sum_{i=1}^{k}\sum_{j=1}^{2^{i-1}}\frac{1}{2^i} = \sum_{i=1}^{k}\frac{1}{2} = k/2.$$

Hence $s_{2^k-1} = k/2 \to \infty$, and $s_n \to \infty$ follows. It is obvious that $a_n < \frac{1}{n}$, and then by the Comparison Test we conclude that $\sum \frac{1}{n}$ diverges as well.

[#15.4*] (a) Either by Comparion/Integral or Comparison Test. By Comparison/Integral Test, we start off by noticing $\frac{1}{\sqrt{n}\log n} \geq \frac{1}{n\log n}$. $f(x) = \frac{1}{x\log x}$ is monotone decreasing for $x \geq 2$. $\sum_{n=2}^{\infty}\frac{1}{n\log n} \geq \int_2^{\infty}\frac{1}{x\log x}dx = \infty$ because $\int\frac{1}{x\log x}dx = \log\log x$. Therefore by Integral Test, $\sum_{n=2}^{\infty}\frac{1}{n\log n}$ diverges, and by Comparison Test $\frac{1}{\sqrt{n}\log n} \geq \frac{1}{n\log n} = \infty$ diverges as well. By Comparison Test alone, we notice $\log n < \sqrt{n}$ for $n \geq 1$, and $\frac{1}{\sqrt{n}\log n} \geq \frac{1}{n}$. Since $\sum \frac{1}{n}$ diverges, $\sum \frac{1}{\sqrt{n}\log n}$ diverges. (b) Either by Comparison or Integral Test. Use Comparison Test we have $\frac{\log n}{n} \geq \frac{1}{n}$ for $n \geq 3$ (assuming log the natural logarithmic, or $n > 10$ if the base 10 logarithmic) and the divergence follows from the divergence of $\sum \frac{1}{n}$. Use Integral Test, we check first that $f(x) = \frac{\log x}{x}$ is monotone decreasing which is the case for $x \geq 3$ since $f'(x) = \frac{1-\log x}{x^2} < 0$. Because $\int_3^{\infty}\frac{\log x}{x}dx = \frac{(\log x)^2}{2}|_3^{\infty} = \infty$, the series $\sum \frac{\log n}{n}$ diverges as well. (c) Use Integral Test on $f(x) = \frac{1}{x\log x(\log\log x)}$. (d) Use Integral Test or Comparison Test. By Integral Test, we use $f(x) = \frac{\log x}{x^2}$ which is monotone decreasing since $f'(x) = \frac{1-2\log x}{x^3} < 0$ for $x \geq 2$. Also $\int\frac{\log x}{x^2}dx = -\frac{\log x}{x} + \int\frac{1}{x^2}dx = -\frac{\log x}{x} - \frac{1}{x}$ using integration by parts. Hence $\int_2^{\infty}\frac{\log x}{x^2}dx = \frac{1+\log 2}{2}$ converges. By Integral Test, the series converges. Use Comparison Test, we note that $\log n < n^q$ for any fixed $0 < q < 1$ and sufficiently large $n > N$. So $\frac{\log n}{n^2} < \frac{n^q}{n^2} = \frac{1}{n^p}$ with $p = 2-q > 1$. Since $\sum\frac{1}{n^p}$ converges for any $p > 1$, we conclude by the Comparison Test that $\sum\frac{\log n}{n^2} < \sum\frac{1}{n^p} < \infty$ converges.

[#15.6*] (a) $a_n = 1/n$. (b) $\sum a_n < +\infty, a_n \geq 0$ implies $a_n^2 \leq a_n$ for all large n since $a_n \to 0$. Comparison Test. (c) $a_n = (-1)^n 1/\sqrt{n}$.

[#17.13*] (a) For any $x \in \mathbb{R}$ and $n \in \mathbb{N}$, there is a rational $r_n \in \mathbb{Q}$ such that $x < r_n < x + 1/n$ by the Archimedean Property. Hence the rational number sequence $r_n \to x$. Also the irrational number sequence $t_n = r_n + \sqrt{2}/n \to x$. Therefore either $\lim f(r_n) = 1 \neq f(x)$ if $x \in \mathbb{R} - \mathbb{Q}$ or $\lim f(t_n) = 0 \neq f(x)$ if $x \in \mathbb{Q}$. f is not continuous in both cases. (b) Similar to (a) when $x \neq 0$. For $x = 0$, we always have $|h(x) - h(0)| = |h(x)| \leq |x|$ to which $\epsilon - \delta$ argument can be easily fashioned.

[#17.14*] If $x \in \mathbb{Q}$, construct an irrational sequence $t_n \in \mathbb{R} - \mathbb{Q}$ in the same way as in #17.13 above so that $t_n \to x$ and $\lim f(t_n) = 0 \neq f(x) = 1/q$ if $x \neq 0$. If $x \in \mathbb{R} - \mathbb{Q}$, $f(x) = 0$ and we claim for every sequence $x_n \to x$, $\lim f(x_n) \to 0$. Otherwise, there is a sequence $x_n \to 0$ but $f(x_n) \nrightarrow 0$, and w.l.o.g we assume $|f(x_n)| \geq \epsilon_0 > 0$ for some constant ϵ_0. This implies then that $x_n \in \mathbb{Q}$ and $f(x_n) =$

$1/q_n \geq \epsilon_0$, which in turn implies $0 < q_n \leq A = 1/\epsilon_0$. Since convergent sequences are bounded, $|p_n/q_n| = |x_n| \leq B$ for some constant B. Hence $|p_n| \leq B|q_n| \leq BA$. Therefore there are only finitely many parings of p_n, q_n for $|p_n| \leq BA, 0 < q_n \leq A$. Therefore the sequence x_n can only take on finitely many values. Since sequence (x_n) converges, x_n must take on a fixed number for all large n and that fixed number is on of the rationals: p_n/q_n with $|p_n| \leq BA$ and $0 < q_n \leq A$. This contradicts to the fact that x_n converges to an irrational number.

[#17.17*] It is obvious that the condition is necessary since it is a special case of the definition that $x_n \to x_0$ implies $f(x_n) \to f(x_0)$. Conversely, assume the contrary that there is a sequence $x_n \to x_0$ with $x_n \in \mathrm{dom}(f)$ but $\lim f(x_n) \nrightarrow f(x_0)$. Then $\exists \epsilon_0 > 0$ so that $\forall N, \exists n \geq N$ with $|f(x_n) - f(x_0)| \geq \epsilon_0$. That is a subsequence can be found so that $|f(x_n) - f(x_0)| \geq \epsilon_0$. Therefore w.o.l.g, we assume x_n is such a subsequence. Then we concluded right away that $x_n \neq x_0$ but $x_n \to x_0$ nonetheless. This contradicts the assumption that we must have $f(x_n) \to f(x_0)$ whenever $x_n \to x_0$ and $x_n \neq x_0$ for all n.

[#18.4*] $f(x) = 1/(x - x_0)$.

[#18.5*] (a) Let $h = f - g$. Then h is continuous as both f and g are continuous. Also $h(a) = f(a) - g(a) \leq 0$ and $h(b) = f(b) - g(b) \geq 0$ by assumption. Then by the Intermediate Value Theorem $h(x_0) = 0$ for some $x_0 \in [a, b]$, implying $f(x_0) = g(x_0)$ as required. (b) Let $g(x) = x$. Then $f(0) \geq 0 = g(0)$ and $f(1) \leq 1 = g(1)$ by the assumption that f maps $[0, 1]$ into $[0, 1]$. Hence the conditions of (a) are satisfied for the given functions f, g and $[a, b] = [0, 1]$.

[#18.10*] Let $g(x) = f(x + 1) - f(x), x \in [0, 1]$. Then g is continuous in $[0, 1]$ as f is continuous in $[0, 2]$. Also, $g(0) = f(1) - f(0), g(1) = f(2) - f(1) = f(0) - f(1)$ by the assumption that $f(2) = f(0)$. Therefore $g(0) = -(f(1) - f(0)) = -g(1)$, implying either $g(0) = g(1) = 0$ or $g(0)$ and $g(1)$ have opposite signs. In the latter case there exists a number $x_0 \in [0, 1]$ such that $g(x_0) = 0$ by IVT. In either cases the same result holds. Therefore with $y_0 = x_0 + 1$ $f(y_0) = f(x_0)$ follows.

[#19.2*] (c) Only. $\forall \epsilon > 0$, let $\delta = \epsilon/4$ s.t. $|x - y| < \delta, x, y \geq 1/2$ implies $|f(x) - f(y)| = |\frac{1}{x} - \frac{1}{y}| = |\frac{y-x}{xy}| \leq |\frac{y-x}{(1/2)(1/2)}| = 4|x - y| < 4\delta = \epsilon$.

[#19.7*] (a) Note first that the continuity of f on $[0, \infty)$ implies the continuity of f on any subset, including $[0, k + 1]$. Since $[0, k + 1]$ is bounded and closed interval of $\mathbb{R}$ f is uniformly continuous on $[0, k + 1]$. We now show that f is uniformly continuous on $[0, \infty)$ by definition. $\forall \epsilon > 0$, $\exists \delta_1 > 0$ s.t. $|x - y| < \delta_1, x, y \in [k, \infty)$ implies $|f(x) - f(y)| < \epsilon$ by the assumption that f is uniformly continuous on $[k, \infty)$. Since f is uniformly continuous on $[0, k+1]$, $\exists \delta_2 > 0$ s.t. $|x - y| < \delta_2, x, y \in [0, k + 1]$ implies $|f(x) - f(y)| < \epsilon$. Let $\delta = \min\{1, \delta_1, \delta_2\}$, we claim that $|x-y| < \delta, x, y \in [0, \infty)$ implies $|f(x)-f(y)| < \epsilon$. To this end all we need to show is that the condition $|x-y| < \delta, x, y \in [0, \infty)$ implies either $x, y \in [0, k+1]$ or $x, y \in [k, \infty)$. Suppose $x, y \in [0, k + 1]$ does not hold. Then either both $x, y \geq k + 1 > k$ or one of x, y is in $[0, k + 1]$ while the other is not. In the former case we have $x, y \in [k, \infty)$. In the latter case, suppose w.o.l.g that $x < k + 1 \leq y$. Then the condition that $|x - y| < \delta \leq 1$ implies that $x = y - (y - x) \geq y - |y - x| \geq y - \delta > k + 1 - 1 = k$. That is, $k < x < y$ and $x, y \in [k, \infty)$ holds as well. (b) Obviously $f(x) = \sqrt{x}$ is continuous in $[0, \infty)$. Because $f'(x) = \frac{1}{2}\frac{1}{\sqrt{x}} \leq \frac{1}{2}\frac{1}{1}$ for $x \geq 1$, hence f is uniformly continuous on $[1, \infty)$. By (a) f is uniformly continuous in $[0, \infty)$.

[#19.9*] (a) $f(x) = x \sin(\frac{1}{x})$ for $x \neq 0$ and $f(0) = 0$ is continuous on $\mathbb{R}$. This follows from the product and composition rules for continuous functions when $x \neq 0$ and the estimate $|f(x) - f(0)| \leq |x|$ at the point 0. (b) Let S be any bounded subset of $\mathbb{R}$. Then $a = \inf S$ and $b = \sup S$ are all finite numbers. Therefore f is uniformly continuous in the bounded, closed interval $[a, b]$ since f is continuous there. Hence f is uniformly continuous on any subset of $[a, b]$ which includes S. (c) Because $f'(x) = \sin(\frac{1}{x}) - \frac{1}{x}\cos(\frac{1}{x})$ for $x \neq 0$ we have $|f'(x)| \leq 1+1 = 2$ for $|x| \geq 1$. Hence f is uniformly continuous in $(-\infty, -1]$ and $[1, \infty)$. Because f is also uniformly continuous in, say $[-2, 2]$, the exactly same argument for #19.7(a) can be used to show f is uniformly continuous in $\mathbb{R}$.

[Notes Supplement On Riemann Integral] Let $U(\{x_i\}) = \sum \sup_{I_i} f \Delta x_i$ be the upper sum of any partition $a = x_0 < x_1 < \cdots < x_n = b, \Delta x_i =$

$x_i - x_{i-1}, i = 1, 2, \ldots, n, I_i = [x_{i-1}, x_i]$. We claim that $\lim_{\Delta x \to 0} U(\{x_i\}) = \ell$ exists where $\Delta x = \max\{\Delta x_i, i = 1, 2, \ldots, n\}$.

Since f is continuous in $[a, b]$, $\exists \bar{x}_i \in [x_{i-1}, x_i]$ such that $f(\bar{x}_i) = \sup_{I_i} f$. Since f is uniformly continuous in $[a, b]$, then $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $|x - y| < \delta, x, y \in [a, b]$ implies $|f(x) - f(y)| < \epsilon$.

We now proceed to prove the claim by first developing a background result. A partition $a = y_0 < y_1 < \ldots y_m = b$ is said to be a *refinement* of a given partition $a = x_0 < x_1 < \cdots < x_n = b$ if $\{x_i\}$ is just a subset of $\{y_j\}$, i.e., $x_i = y_{j_i}$ for some j_i and for all $i = 0, 1, 2, \ldots, n$. Then the difference between the corresponding upper sums $U(\{x_i\}) - U(\{y_j\})$ has the following properties.

Either a subinterval $[x_{i-1}, x_i]$ contains no refinement points y_j with $x_i = y_{j_i}$ for some j_i and $x_{i-1} = y_{j_i - 1}$. In this case the corresponding summands $\sup_{I_i} f \Delta x_i$ and $\sup_{J_{j_i}} f \Delta y_{j_i}$ are identical and cancel out each other in the difference $U(\{x_i\}) - U(\{y_j\})$.

Or a subinterval $[x_{i-1}, x_i]$ contains some refinement points $x_{i-1} = y_j < y_{j+1} < \cdots < y_{j+k} = x_i$ for some $k > 1$. In this case the summand $\sup_{I_i} f \Delta x_i$ corresponds to the subsum $\sum_{l=1}^{k} \sup_{J_{j+l}} f \Delta y_{j+l}$ with $J_{j+l} = [y_{j+l-1}, y_{j+l}]$. Breaking up $[x_{i-1}, x_i]$ according to its refinement $x_{i-1} = y_j < y_{j+1} < \cdots < y_{j+k} = x_i$, the corresponding difference in absolute value $|\sup_{I_i} f \Delta x_i - \sum_{l=1}^{k} \sup_{J_{j+l}} f \Delta y_{j+l}|$ becomes

$$|\sum_{l=1}^{k} (\sup_{I_i} f - \sup_{J_{j+l}} f) \Delta y_{j+l}| < \sum_{l=1}^{k} \epsilon \Delta y_{j+l} \le \epsilon \Delta x_i$$

if $\Delta x_i = x_i - x_{i-1} < \delta$ by the uniformly continuity since $\sup_{I_i} f = f(\bar{x}_i)$ and $\sup_{J_{j+l}} f = f(\bar{y}_{j+l})$ with $\bar{x}_i, \bar{y}_{j+l} \in [x_{i-1}, x_i] \implies |\bar{x}_i - \bar{y}_{j+l}| \le x_i - x_{i-1} < \delta$. Hence the upper sum difference in absolute value $|U(\{x_i\}) - U(\{y_j\})|$ on a whole is bounded above by $\epsilon \sum_{i=1}^{n} \Delta x_i = \epsilon(b - a)$ for any refinement of partition $\{x_i\}$ and $\Delta x \le \delta$.

We are now ready to prove the claim $\lim_{\Delta x \to 0} U(\{x_i\}) = \ell$. As we did in class we first show that the sequence $U_n = U(\{x_i\})$ in regular partition $x_i = a + i\Delta x, \Delta x = (b - a)/n$ has a limit. We do this by showing that $\{U_n\}$ is a Cauchy sequence. In fact, for $(b - a)/N < \delta$ or $N > (b - a)/\delta$ and any $m, n > N$, the partition for U_{mn} is a refinement for both partitions of U_n and U_m because the partition points of U_n satisfy $x_i = a + i\frac{b-a}{n} = a + im\frac{b-a}{nm}$ which is a partition point of U_{mn} for each i and similarly for U_m. By what we have just proved above, $|U_n - U_m| = |U_n - U_{mn} + U_{mn} - U_m| \le |U_n - U_{mn}| + |U_{mn} - U_m| < 2\epsilon(b - a)$ since $\Delta x = (b - a)/n$ for U_n and $\Delta x = (b - a)/m$ for U_m are both less than $(b - a)/N < \delta$ for $m, n > N$. This shows U_n is Cauchy and $\lim U_n = \ell$ follows.

Finally we prove $\lim_{\Delta x \to 0} U(\{x_i\}) = \ell$ for all partition. Assume on the contrary that it is false, then a sequence of upper Riemann sums $U(\{x_i^k\}), k = 1, 2, \ldots$ can be found such that $|U(\{x_i^k\}) - \ell| > \epsilon_0$ for some fixed number ϵ_0 even though $\Delta x^k = \max\{\Delta x_i^k, i = 1, 2, \ldots n_k\} \to 0$ as $k \to 0$. Because $U_n \to \ell$, we have $N_0 > 0$ such $n > N_0$ implies $|U_n - \ell| < \epsilon_0/2$ and thus $|U(\{x_i^k\}) - U_n| = |U(\{x_i^k\}) - \ell - (U_n - \ell)| \ge |U(\{x_i^k\}) - \ell| - |U_n - \ell| > \epsilon_0/2$ for $n > N_0$ and all k. This has to be a contradiction for the following reasons. For each regular nth partition and any given one in $\{x_i^k\}$, putting all these points together to form a refinement partition for both U_n and $U(\{x_i^k\})$ and denote the refinement upper sum by U_n^k. Then when $\Delta x^k, (b - a)/n < \delta$, we have $|U(\{x_i^k\}) - U_n| = |U(\{x_i^k\}) - U_n^k + U_n^k - U_n| \le |U(\{x_i^k\}) - U_n^k| + |U_n^k U_n| < 2\epsilon(b - a) < \epsilon_0/2$ since ϵ is arbitrary. This completes the proof.

[#20.11*] Follow the hints.

[#20.16*(c)] (a) The case with $L_1 = +\infty$ must imply $L_2 = \infty = L_1$ because $\forall M > 0$, $\exists \delta > 0$ s.t. $a < x < a + \delta$ implies $M < f_1(x) \le f_2(x)$. Similarly the case with $L_2 = -\infty$ implies $L_1 = L_2 = -\infty$. In both cases $L_1 \le L_2$ follows. For the remaining case we only have $L_1, L_2 \in \mathbb{R}$. In this case $\forall \epsilon > 0$, $\exists \delta_1 > 0, \delta_2 > 0$ s.t. $a < x < a + \delta_1 \implies |f_1(x) - L_1| < \epsilon$ and $a < x < a + \delta_2 \implies |f_2(x) - L_2| < \epsilon$. Then for $a < x < a + \delta$ with $\delta = \min\{\delta_1, \delta_2\}$ we have

$$\begin{aligned} L_1 &= f_1(x) + (L_1 - f_1(x)) \\ &< f_1(x) + \epsilon \\ &\le f_2(x) + \epsilon = f_2(x) - L_2 + L_2 + \epsilon \\ &< \epsilon + L_2 + \epsilon = L_2 + 2\epsilon \end{aligned}$$

implying $L_1 \le L_2$ since $\epsilon > 0$ is arbitrary. (b) No. For example $f_1(x) = x^2, f_1(x) = x, x \in (0, 1)$.

[#20.17*] Follow the hints.

[#20.20*(c)] By definition for any sequence $x_n \to a$ with $x_n \in S \subset \mathbb{R}$ we have $f_1(x_n) \to +\infty$ and $f_2(x_n) \to L_2 < 0$. Since $-f_2(x_n) \to -L_2 > 0$ by Theorem 9.9 $\lim f_1(x_n)(-f_2(x_n)) = +\infty$. By #9.10(c), $\lim f_1(x_n)f_2(x_n) = \lim(-1)[f_1(x_n)(-f_2(x_n))] = -\infty$. Alternatively, you can prove it directly. By definition, for any sequence $x_n \to a$ with $x_n \in S$ we have $f_1(x_n) \to +\infty$ and $f_2(x_n) \to L_2 < 0$. That is, if we let $A = \max\{L_2/2, -1\} < 0$, then $\forall M > 0$, $\exists N_1$ s.t. $n > N_1$ implies $f_1(x_n) > M/|A|$ and $\exists N_2$ s.t. $n > N_2$ implies $f_2(x_n) < A = \max\{L_2/2, -1\} < 0$. Let $N = \max\{N_1, N_2\}$. then $n > N$ implies $f_1(x_n)f_2(x_n) < f_1(x_n)A < \frac{M}{|A|}A = -M$ because $A = -|A|$. This implies by definition that $\lim_1(x_n)f_2(x_n) = -\infty$. Since this holds for all sequences $x_n \to a, x_n \in S$ we have by definition that $\lim_{x\to a^S} f_1(x)f_2(x) = -\infty$.

[#23.2*] (a) $a_n^{1/n} = n^{1/2n} = \exp(\frac{\ln n}{2n}) \to \exp(0) = 1$. $R = 1/\limsup a_n^{1/n} = 1/1 = 1$. At $x = \pm 1$, $\sum \sqrt{n}x^n$ diverge because $\sqrt{n}x^n = \sqrt{n}$, or $(-1)^n\sqrt{n}$ and none of which goes to 0 as required for convergent series. Interval of convergence $(-1, 1)$. (b) $a_n^{1/n} = 1/n^{1/\sqrt{n}} = 1/\exp(\frac{\ln n}{\sqrt{n}}) \to 1/\exp(0) = 1$ and $R = 1/1/\limsup a_n^{1/n} = 1$. At $x = \pm 1$, $|a_nx^n| = 1/n^{\sqrt{n}} \le 1/n^2$ for $n \ge 4$. Hence both sequences $\sum a_nx^n$ at $x = \pm 1$ converge by dominant comparison test. Interval of convergence $[-1, 1]$. (c) Notice $c_k = 1$ if $k = n!$ for some n and 0 otherwise. So $c_k^{1/k} = 0$ or 1 for infinitely many k and $\limsup c_k^{1/k} = 1$ and $R = 1/\limsup c_k^{1/k} = 1$. At $x = \pm 1$, $|x^{n!}| = 1 \nrightarrow 0$ and $\sum x^{n!}$ diverges at the end points by the necessary criterion for converging series. Interval of convergence $(-1, 1)$. (d) $c_n^{1/n} = 0$ if n is even and $c_n^{1/n} = 3^{k/(2k+1)}/k^{1/(4k+2)}$ for $n = 2k + 1$ is odd. Thus $\limsup c_n^{1/n} = \lim_{k\to\infty} 3^{k/(2k+1)}/k^{1/(4k+2)} = 3^{1/2}/1 = \sqrt{3}$. $R = 1/\limsup c_n^{1/n} = 1/\sqrt{3}$. At $x = 1/\sqrt{3}$, $\sum a_nx^n = \sum \frac{3^n}{\sqrt{n}}(\frac{1}{\sqrt{3}})^{2n+1} = \frac{1}{\sqrt{3}}\sum\frac{1}{\sqrt{n}}$ which diverges as a p-series with $p = 1/2 < 1$. At $x = -1/\sqrt{3}$, $\sum a_nx^n = \sum \frac{3^n}{\sqrt{n}}(-\frac{1}{\sqrt{3}})^{2n+1} = -\frac{1}{\sqrt{3}}\sum\frac{1}{\sqrt{n}}$ which diverges for the same reason.

[#23.4*] (a) $(a_n)^{1/n} = 6/5$ if n is even and $2/5$ if n is odd. So $\limsup(a_n)^{1/n} = 6/5, \liminf(a_n)^{1/n} = 2/5$. Similarly $|a_{n+1}/a_n| = \frac{6}{5}3^n$ when n is odd and $\frac{2}{5}(\frac{1}{3})^n$ when n is even. Hence $\limsup|a_{n+1}/a_n| = \infty$ and $\liminf|a_{n+1}/a_n| = 0$. (b) Both diverge because the necessary condition that $a_n \to 0$ does not satisfied because of the fact that $\limsup(a_n)^{1/n} = 6/5 > 1$. (c) The radius of convergence is $R = 1/\limsup(a_n)^{1/n} = 5/6$. At $x = 5/6$, $\sum a_nx^n = \sum(\frac{4+2(-1)^n}{6})^n$ diverges because $(\frac{4+2(-1)^n}{6})^n \nrightarrow 0$. Similarly, at $x = -5/6$, $\sum a_nx^n = \sum(-1)^n(\frac{4+2(-1)^n}{6})^n$ diverges because $(-1)^n(\frac{4+2(-1)^n}{6})^n \nrightarrow 0$.

[#23.8*] (a) $f_n(x) = \sum_0^n x^k = \frac{1-x^{n+1}}{1-x}$ if $x \in [0, 1)$ and $f_n(1) = n+1$. Thus $f_n \to \frac{1}{1-x}$ point wise in $[0, 1)$ but not at $x = 1$. (b) Since f_n does not converge point wise on $[0, 1]$, it does not converge uniformly on $[0, 1]$.

[#23.16*] (a) $f_n(x) = \frac{nx}{1+nx^2} \to 0$ if $x = 0$ and $1/x$ if $x > 0$. So $f(x) = 0$ if $x = 0$ and $1/x$ if $x > 0$. (b) The convergence is not uniform in $[0, 1]$ because otherwise f would be continuous in $[0, 1]$ as each f_n is continuous. Alternatively, we can show directly by definition that $f_n \to f$ not uniformly. In fact, for $\epsilon_0 = 1$ and $\forall n$, solving $|f_n(x) - f(x)| = 1$ is equivalent to $|\frac{nx}{1+nx^2} - \frac{1}{x}| = \frac{1}{x+nx^3} = 1$, equivalent to $p(x) = nx^3 + x - 1 = 0$, for which it must have a solution x_n in $[0, 1]$ by Intermediate Value Theorem since $p(0) = -1 < 0$ and $p(1) = n > 0$. This shows $\exists x_n \in (0, 1)$ s.t. $|f_n(x_n) - f(x_n)| = 1$ for all $n \ge 1$ and $\lim f_n = f$ is not uniform. (c) Because $|f_n(x) - f(x)| = \frac{1}{x+nx^3} \le 1/(1 + n)$ for all $x \ge 1$, $\lim f_n = f$ uniformly on $[1, \infty)$. In fact, $\forall\epsilon > 0$, let $N = 1/\epsilon$ then for $n > N$ and $x \in [1, \infty)$ we have $|f_n(x) - f(x)| = \frac{1}{x+nx^3} \le 1/(1 + n) < 1/N = \epsilon$.

[#23.17*] By assumption, $\forall\epsilon > 0$, $\exists N_1 > 0$ s.t. $n > N_1$ implies $|f_n(y) - f(y)| < \epsilon/2$ for all $y \in [a, b]$, in particular, $|f_n(x_n) - f(x_n)| < \epsilon/2$ for any sequence $x_n \in [a, b]$ with $x_n \to x$. Since the uniform limit of continuous functions is continuous and that $x_n \to x$, $\exists N_2 > 0$ s.t. $|f(x_n) - f(x)| < \epsilon/2$ for $n > N_2$. Let $N = \max\{N_1, N_2\}$. Then $n > N$ implies that $|f_n(x_n) - f(x)| = |f_n(x_n) - f(x_n) + f(x_n) - f(x)| \le |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \epsilon/2 + \epsilon/2 = \epsilon$. This shows, by definition, that $\lim f_n(x_n) = f(x)$.

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SOLUTIONS TO SELECTED EXAM II PROBLEMS

[#2] ($\Longrightarrow$) Suppose the contrary: $\exists N_0 > 0$ s.t. $\forall \delta > 0$, in particular, $\delta_n = 1/n$, $\exists x_n \in S$ with $|x_n - a| < 1/n$ but $f(x_n) \le N_0$. This contradicts the condition that for every sequence $x_n \in S$ with $x_n \to a$ we have $f(x_n) \to +\infty$. ($\Longleftarrow$) By the assumption, $\forall M > 0, \exists \delta > 0$ s.t. $|x - a| < \delta,\ x \in S$ implies $f(x) > M$. Now for any sequence $x_n \in S,\ x_n \to a,\ \exists N > 0$ s.t. $n > N$ implies $|x_n - a| < \delta$. Hence $f(x_n) > M$ follows, showing $\lim f(x_n) = +\infty$.

[#3] (a) $a_n^{1/n} = 3^{\frac{(-1)^n}{n} - 1} \to 3^{0-1} = 1/3$ implying $\limsup a_n^{1/n} = \liminf a_n^{1/n} = \lim a_n^{1/n} = 1/3$. $|\frac{a_{n+1}}{a_n}| = 3^{(-1)^{n+1} - n - 1}/3^{(-1)^n - n} = 3^{(-1)^{n+1} - (-1)^n - 1} = 3$ if n is odd and $= 1/27$ if n is even. So $\limsup \frac{a_{n+1}}{a_n} = 3$, $\liminf \frac{a_{n+1}}{a_n} = 1/27$. (b) $c_{2n} = 0, c_{2n+1} = 3^{(-1)^n - n}$. $(c_{2n})^{1/2n} \to 0$ and $(c_{2n+1})^{1/2n+1} = 3^{\frac{(-1)^n}{2n+1} - \frac{n}{2n+1}} \to 3^{0-1/2}$. Hence $\limsup (c_n)^{1/n} = 1/\sqrt{3}$ and the radius of convergence $R = 1/\limsup (c_n)^{1/n} = \sqrt{3}$. (c) At $x = \sqrt{3}$, $\sum 3^{(-1)^n - n}(\sqrt{3})^{2n+1} = \sum 3^{(-1)^n - n} 3^{n+1/2} = \sum 3^{(-1)^n + 1/2} = +\infty$ which diverges because $3^{(-1)^n + 1/2} \nrightarrow 0$ not satisfying the necessary condition for convergence. Similarly, at $x = -\sqrt{3}$, $\sum 3^{(-1)^n - n}(-\sqrt{3})^{2n+1} = -\sum 3^{(-1)^n + 1/2} = -\infty$ diverges for the same reason. Interval of convergence: $(-\sqrt{3}, \sqrt{3})$.

[#5] (a) $f_n(x) = |x|^n \to f(x) = \begin{cases} 0, & |x| < 1 \\ 1, & |x| = 1 \end{cases}$. Since f is not continuous while f_n are, $f_n \to f$ only pointwise not uniform. (b) Yes, because $f(x) = 0, x \in (a, b)$ with $-1 < a < b < 1$, and therefore $x \in (a, b)$ implies $|f_n(x) - f(x)| = |x^n| \le \max\{|a|^n, |b|^n\} \to 0$ independent of x since $|a|, |b| < 1$.

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[#25.3*] (a) Consider $|f_n(x) - \frac{1}{2}| = |\frac{n + \cos x}{2n + \sin^2 x} - \frac{1}{2}| = |\frac{2\cos x - \sin^2 x}{4n + 2\sin^2 x}| \le \frac{3}{4n}$. (b) Use Theorem 25.2.

[#25.8*] Since $|a_n|^{1/n} = 1/(n^{2/n} 2) \to 1/2$, the radius of convergence is $R = 1/(1/2) = 2$. Because for $|x| \le 2$, $|\frac{x^n}{n^2 2^n}| \le \frac{|x|^n}{n^2 2^n} \le \frac{2^n}{n^2 2^n} \le \frac{1}{n^2}$. Since $\sum \frac{1}{n^2} < \infty$ converges by the p-series Test, with $p = 2 > 1$, $\sum \frac{x^n}{n^2 2^n}$ converges uniformly on $|x| \le 2$ by the Dominant Convergence Theorem. Therefore the sum $f(x) = \sum \frac{x^n}{n^2 2^n}$ is continuous in $[-2, 2]$.

[#25.12*] Let $f_n = \sum_{k=0}^n g_k$. Then that g_k is continuous in $[a, b]$ implies f_n is continuous in $[a, b]$. Also $\sum g_k$ converges uniformly on $[a, b]$ iff $f_n \to g = \sum g_k$ uniformly. Then by Theorem 25.2 $\int_a^b g(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx = \lim_{n\to\infty} \int_a^b \sum_{k=0}^n g_k(x)dx = \lim_{n\to\infty} \sum_{k=0}^n \int_a^b g_k(x)dx = \sum_{k=0}^\infty \int_a^b g_k(x)dx$.

[#25.15*] (a) Assume $f_n \to 0$ is not uniform. Then $\exists \epsilon_0 > 0$ and $\forall n, \exists x_n \in [a, b]$ so that $|f_n(x_n)| \ge \epsilon_0$. Since $\{f_n(x)\}$ is non-increasing in n we have for $k \ge n$ that $f_n(x) \ge f_k(x) \to 0$ as $k \to \infty$ implying $f_n(x) \ge 0$ for all n. Therefore $f_n(x_n) \ge \epsilon_0$. Next by Weierstrass Theorem, $\{x_n\}$ has a convergent subsequence $x_{n_k} \to x_0 \in [a, b]$, and hence $f_{n_k}(x_{n_k}) \ge \epsilon_0$. So w.o.l.g we can assume $x_n \to x_0$ and $f_n(x_n) \ge \epsilon_0$ as originally stated. Because $\{f_n(x)\}$ is non-increasing in n, we have $f_m(x) \ge f_n(x)$ for all $m \le n$ and all $x \in [a, b]$, in particular for $x = x_n$. That is $f_m(x_n) \ge f_n(x_n) \ge \epsilon_0$ for all $m \le n$. This will give rise to a contradiction. In fact, because $f_m(x_0) \to 0$ as $m \to \infty$ by the pointwise convergence assumption, $\exists M > 0$ s.t. $m > M$ implies $f_m(x_0) < \epsilon_0/2$. For this M, since f_{M+1} is continuous, $\exists N_1$ s.t. $n > N_1$ implies $|f_{M+1}(x_n) - f_{M+1}(x_0)| < \epsilon_0/2$. Putting these two results together we have for $n > \max\{N_1, M\}$ the following $f_{M+1}(x_n) = f_{M+1}(x_n) - f_{M+1}(x_0) + f_{M+1}(x_0) < \epsilon_0/2 + \epsilon_0/2 = \epsilon_0$ contradicting to the statement that $f_{M+1}(x_n) \ge \epsilon_0$ since $n \ge M + 1$. (b) Let $g_n(x) = f_n(x) - f(x)$. Because f is continuous by assumption g_n is continuous. Because $\{f_n(x)\}$ is nonincreasing in n $\{g_n(x)\}$ is nonincreasing in n for each $x \in [a, b]$. Also $g_n \to 0$ pointwise. So the conditions of (a) are satisfied for g_n, and $g_n \to 0$ uniformly follows, implying $f_n \to f$ uniformly as wanted.

[#26.6*] Let $s(x) = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)!} x^{2n+1}, c(x) = \sum_{n=0}^\infty \frac{(-1)^n}{(2n)!} x^{2n}$. (a) Let $s(x) = \sum c_k x^k$ then $(c_k)^{1/k} = \begin{cases} 0, & k = 2n \\ (1/(2n+1)!)^{1/(2n+1)}, & k = 2n+1. \end{cases}$ So $\limsup |c_k|^{1/k} = \lim (1/(2n+1)!)^{1/(2n+1)} = 0$ because $\lim (1/n!)^{1/n} = \lim \frac{1/(n+1)!}{1/n!} = \lim \frac{1}{n+1} = 0$. Therefore the radius of convergence for $s(x)$ is $R = 1/\limsup |c_k|^{1/k} = \infty$ and $s(x)$ converges for all

$x \in \mathbb{R}$. Then by the derivative theorem of power series we have $s'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(2n+1)x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} = c(x)$. Exactly the same argument shows $c'(x) = -s(x)$. (b) By (a), $(s^2(x) + c^2(x))' = 2ss' + 2cc' = 2sc + 2c(-s) \equiv 0$. (c) Because $(s^2(x) + c^2(x))' \equiv 0$, $s^2(x) + c^2(x) \equiv$constant and $s^2(x) + c^2(x) \equiv s^2(0) + c^2(0) = 0 + 1 = 1$ follows.

[#26.7*] No, because all power series with positive radius of convergence are differentiable at $x = 0$ whereas $f(x) = |x|$ is not differentiable at $x = 0$.

[#26.8*] (a) Since $\sum_{n=0}^{\infty} y^n = 1/(1-y)$ for $|y| < 1$, for $|x| < 1$ and $y = -x^2$ we have by direct substitution that $\sum_{n=0}^{\infty}(-1)^n x^{2n} = \sum_{n=0}^{\infty}(-x^2)^n = 1/(1-(-x^2)) = 1/(1+x^2)$. (b) Because $1/(1+x^2) = \sum_{n=0}^{\infty}(-1)^n x^{2n}$ holds for $x \in (-1,1)$, by the integral theorem of power series we have for all $x \in (-1,1)$ that $\arctan x = \arctan x - \arctan 0 = \int_0^x \frac{1}{1+t^2}dt = \int_0^x \sum_{n=0}^{\infty}(-1)^n t^{2n}dt = \sum_{n=0}^{\infty}(-1)^n \int_0^x t^{2n}dt = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^{2n+1}$. (c) At $x = 1$, the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ converges by the Alternating Test since $a_n = 1/(2n+1) \to 0$ and $a_n < a_{n-1}$. Thus by Abel's Theorem the power series $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^{2n+1}$ converges and is continuous in $(-1,1]$. Because $\arctan x$ is continuous, we have $\pi/4 = \arctan 1 = \lim_{x\to 1^-} \arctan x = \lim_{x\to 1^-} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$. Thus $\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^{2n+1}$ holds in $(-1,1]$ and $\pi = 4\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ follows. (d) Exactly the same argument of (c) applies to $x = -1$.

[#27.3*] (a) Assume on the contrary that there is a sequence of polynomials $p_n(x)$ converging uniformly to $\sin x$ on $\mathbb{R}$, then there is an $N > 0$ such that $n > N$ implies $|p_n(x) - \sin x| < 1$. $p_n(x)$ cannot be a constant polynomial for otherwise, we would have $2 = |\sin(\pi/2) - \sin(-\pi/2)| = |\sin(\pi/2) - p_n + p_n - \sin(-\pi/2)| \le |\sin(\pi/2) - p_n| + |p_n - \sin(-\pi/2)| < 1 + 1 = 2$, a contradiction. Since $p_n(x)$ is not a constant polynomial, it is unbounded, so is $|p_n(x) - \sin x|$, contradicting to the assumption that $|p_n(x) - \sin x| < 1$ for all $x \in \mathbb{R}$. (b) Because for any polynomial $\sum_{k=0}^{n} a_k x^k$ we have for $x > 0$ that $e^x > x^{n+1}/(n+1)!$ and $|e^x - \sum_{k=0}^{n} a_k x^k| \ge x^{n+1}/(n+1)! - \sum_{k=0}^{n} |a_k| x^k > 1$ for all sufficiently large $x > 0$. Hence it is not possible to have a sequence of polynomials to converge uniformly to e^x.

[#27.6*] Because the Bernstin polynomials are continuous functions on $[0,1]$.

[#28.2*] Use definition.

[#28.4*] (a) For $x \ne 0$, $f(x) = x^2\sin(\frac{1}{x})$ is continuously differentiable by product and chain rules, and $f'(x) = 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x})$. (b) By definition,

$$\begin{aligned} f'(0) &= \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x\to 0} \frac{x^2\sin(1/x) - 0}{x} = \lim_{x\to 0} x\sin(1/x) = 0 \end{aligned}$$

because $\sin(1/x)$ is bounded and $x \to 0$ in the product. (c) Since $\lim_{x\to 0} 2x\sin(1/x) = 0$ and $\lim_{x\to 0}\cos(1/x)$ does not exist, $\lim x \to 0 f'(x) = \lim_{x\to 0}(2x\sin(1/x) - \cos(1/x))$ does not exist. Therefore $\lim_{x\to 0} f'(x) = f'(0) = 0$ does not hold, and f' is not continuous at $x = 0$ by definition.

[#28.6*] Similar to #28.4 above.

[#29.2*] Let $f(x) = \cos x$. f is differentiable everywhere and $f'(x) = -\sin x$. Since f is continuous on $[x,y]$ (or $[y,x]$ if $y < x$) and differentiable on (x,y) (or (y,x) respectively), then by MVT, there exists a z between x, y such that

$$\frac{\cos x - \cos y}{x - y} = \frac{f(x) - f(y)}{x - y} = f'(z) = -\sin z.$$

Hence $|\cos x - \cos y| = |-\sin z(x-y)| = |\sin z||x - y| \le |x - y|$ follows.

[#29.5*] By definition,

$$\begin{aligned} |f'(x)| &= \lim_{y\to x} \left|\frac{f(y) - f(x)}{y - x}\right| \\ &\le \lim_{y\to x} \frac{|y - x|^2}{|y - x|} = \lim_{y\to x} |y - x| = 0. \end{aligned}$$

Therefore $f'(x) = 0$ for all x and $f \equiv$constant.

[#29.10*] Let $g(x) = x^2\sin(1/x), x \ne 0$ and $g(0) = 0$. Then $f(x) = g(x) + x/2$. From #28.4 g is differentiable everywhere with $g'(0) = 0$ but the derivative g' is continuous everywhere except at $x = 0$. Then by the summation theorem of derivatives we have $f'(0) = g'(0) + 1/2 = 1/2 > 0$. (b) For any interval (a,b), $a < 0 < b$ about 0, there exists an integer

n such that $0 < x_n = \frac{1}{2n\pi} < b$. At the point x_n, $f'(x_n) = 0 - 1 + 1/2 = 1/2$. Since f' is continuous at x_n, $\exists \delta > 0$ s.t. $|x - x_n| < \delta \Longrightarrow |f'(x) - f'(x_n)| < 1/4$ which implies $f'(x) < f'(x_n) + 1/4 = -1/4 < 0$ for all $x \in (x_n - \delta, x_n + \delta) \subset (a, b)$. Therefore f is strictly decreasing in the subinterval $(x_n - \delta, x_n + \delta)$. (c) Even though $f'(0) > 0$ function f does not satisfy the condition of Corollary 29.7(i) in any neighborhood of 0, hence the result does not apply. Moreover, this example shows the result can be false if the sufficient conditions are not met.

[#29.13*] Let $h(x) = g(x) - f(x)$. Then $h(0) = g(0) - f(0) = 0$ and $h'(x) = g'(x) - f'(x) \geq 0$ for all $x \in \mathbb{R}$. Therefore h is increasing on $\mathbb{R}$. In particular, for $x \geq 0$, $h(x) \geq h(0) = 0$, implying $g(x) - f(x) \geq 0$ and $g(x) \geq f(x)$ for all $x \geq 0$ follows.

[#29.18*] By MVT, there exists an x_n between s_{n-1} and s_n for all $n \in \mathbb{N}$ such that

$$\left|\frac{s_{n+1} - s_n}{s_n - s_{n-1}}\right| = \left|\frac{f(s_n) - f(s_{n-1})}{s_n - s_{n-1}}\right| = |f'(x_n)| \leq a < 1.$$

Thus $|s_{n+1} - s_n| \leq a|s_n - s_{n-1}| \leq a^n |s_1 - s_0|$. Now for any $m \geq n$, let $m = n + k$ with $k \geq 0$. We have

$$\begin{aligned}
|s_m - s_n| &= |s_{n+k} - s_{n+k-1} + s_{n+k-1} - \cdots \\
&\qquad - s_{n+1} + s_{n+1} - s_n| \\
&\leq |s_{n+k} - s_{n+k-1}| + \cdots |s_{n+1} - s_n| \\
&\leq a^{n+k-1}|s_1 - s_0| + a^{n+k-2}|s_1 - s_0| + \cdots \\
&\qquad + a^n |s_1 - s_0| \\
&= (a^{n+k-1} + a^{n+k-2} + \cdots + a^n)|s_1 - s_0| \\
&= a^n \frac{1 - a^k}{1 - a}|s_1 - s_0| \leq \frac{a^n}{1 - a}|s_1 - s_0|
\end{aligned}$$

since $0 \leq a < 1$. If $s_1 = s_0$, then $|s_m - s_n| = 0 < \epsilon$ for any ϵ and $\{s_n\}$ is clearly a Cauchy sequence. Otherwise if $s_1 \neq s_0$ then $\forall \epsilon > 0$ let $N = \max\{1, \left(\ln \frac{\epsilon(1-a)}{|s_1 - s_0|}\right) / \ln a\}$, then $m \geq n > N$ implies $|s_m - s_n| \leq \frac{a^n}{1-a}|s_1 - s_0| < \frac{a^N}{1-a}|s_1 - s_0| = \epsilon$. Hence $\{s_n\}$ is Cauchy and $\lim_{n\to\infty} s_n = s^* \in \mathbb{R}$ exists, with s^* satisfying $f(s^*) = s^*$.

[#31.1*] Let $f(x) = \cos x$. Then for $k = 0, 1, 2, \ldots$ and $x = 0$, $f^{(4k)}(x) = \cos x = 1, f^{(4k+1)}(x) = -\sin x = 0, f^{(4k+2)}(x) = -\cos x = -1, f^{(4k+3)}(x) = -\sin x = 0$. Hence the Taylor series is $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j}}{(2j)!}$. Because $|f^{(n)}(x)| \leq 1$, by Taylor's Theorem, $|R_n(x)| \leq \frac{|x|^n}{n!} \to 0$ as $n \to \infty$ for every $x \in \mathbb{R}$. Hence the Taylor series of $\cos x$ converges to itself and $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$.

[#31.3*] (a) By induction there exist polynomials $p_n(x)$ of degree $3n$ such that $g^{(n)}(x) = p_n(\frac{1}{x}) e^{-1/x^2}$ for $x \neq 0$ and $n \geq 1$. For $n = 1$, it is straightforward to verify that $g'(x) = \frac{2}{x^3} e^{-1/x^2}$ for $x \neq 0$. Assume the case for n, and consider $g^{(n+1)}(x) = (g^{(n)}(x))' = (p_n(\frac{1}{x}) e^{-1/x^2})' = p_n'(\frac{1}{x})(-\frac{1}{x^2}) e^{-1/x^2} + p_n(\frac{1}{x}) \frac{2}{x^3} e^{-1/x^2} = p_{n+1}(\frac{1}{x}) e^{-1/x^2}$ with $p_{n+1}(x)$ clearly a polynomial of degree $3(n+1)$. To show $g^{(n)}(0) = 0$ we use induction again. $g'(0) = \lim_{x\to 0} \frac{g(x) - g(0)}{x - 0} = \lim_{x \to 0} \frac{e^{-1/x^2}}{x} = \lim_{y\to\infty} \frac{y}{e^{y^2}}$ with $y = 1/x$. Since $e^{y^2} = \sum \frac{y^{2n}}{n!} \geq \frac{y^{2n}}{n!}$ for any n, we have $\lim_{y\to\infty} \frac{y}{e^{y^2}} < \frac{yn!}{y^{2n}} = 0$. Similarly, assume $g^{(n)}(0) = 0$ we can show $g^{(n+1)}(0) = \lim_{x\to 0} \frac{g^{(n)}(x) - g^{(n)}(0)}{x - 0} = \lim_{x\to 0} p_n(1/x) e^{1/x^2}/x = \lim_{y\to\infty} y p_n(y)/e^{y^2} \leq \lim_{y\to\infty} y p_n(y) k!/y^{2k} = 0$ as k can be chosen so that $2k > 3n + 1$, with the latter to be the degree of $y p_n(y)$. (b) Since $g^{(n)}(0) = 0$, the Taylor series is $f(x) = \sum \frac{g^{(n)}(0)}{n!} x^n \equiv 0 = g(0)$ which agrees with $g(x)$ only at $x = 0$.

[#31.6*] (a) Let M be defined as $f(x) = f(0) + \sum_{k=1}^{n-1} \frac{f^{(k)}(0)}{k!} x^k + \frac{M}{n!} x^n$ for $x \neq 0$, that is $M = n!(f(x) - f(0) - \sum_{k=1}^{n-1} \frac{f^{(k)}(0)}{k!} x^k)/x^n$. Define for $t \in [0, x]$, $F(t) = f(t) + \sum_{k=1}^{n-1} \frac{(x-t)^k}{k!} f^{(k)}(t) + M \frac{(x-t)^n}{n!}$. Then $F(t)$ is continuous in $[0, x]$ and differentiable in $(0, x)$ because $f^{(n)}$ exists in $|x| < R$. In addition, $F(0) = f(0) + \sum_{k=1}^{n-1} \frac{f^{(k)}(0)}{k!} x^k + \frac{M}{n!} x^n = f(x)$ by the definition for M, and $F(x) = f(x)$ obviously. Hence $F(0) = F(x)$, and Rolle's Theorem, there is a $y \in (0, x)$ such that $F'(y) = 0$. Since $F'(t) = f'(t) + \sum_{k=1}^{n-1} (-\frac{(x-t)^{(k-1)}}{(k-1)!}) f^{(k)}(t) + \sum_{k=1}^{n-1} \frac{(x-t)^k}{k!} f^{(k+1)}(t) - M \frac{(x-t)^{(n-1)}}{(n-1)!}$. Changing the summation index by $k - 1 = i$ for the first sum and renaming $k = i$ for the second, regrouping, simplifying by cancellation, then we have $F'(t) = f'(t) +$

$\sum_{i=0}^{n-2}(-\frac{(x-t)^i}{i!})f^{(i+1)}(t) + \sum_{i=1}^{n-1}\frac{(x-t)^i}{i!}f^{(i+1)}(t) - M\frac{(x-t)^{(n-1)}}{(n-1)!} = \frac{(x-t)^{(n-1)}}{(n-1)!}f^{(n)}(t) - M\frac{(x-t)^{(n-1)}}{(n-1)!} = \frac{(x-t)^{(n-1)}}{(n-1)!}(f^{(n)}(t) - M)$. Since $y \neq x$, $F'(y) = \frac{(x-y)^{(n-1)}}{(n-1)!}(f^{(n)}(y) - M) = 0$ iff $f^{(n)}(y) - M = 0$.