Math 423/823 Spring 2014 Exam 1

Name: ______________________________ **Score:** ________

Instructions: You must show supporting work to receive full and partial credits. No text book, notes, or formula sheets allowed.

1. (10 pts) Find all values of $(1-i\sqrt{3})^{1/2}$ in the Cartesian coordinate form.

2. (15 pts) Show that if $z_1 = i$ and $z_2 = i - 1$, then
$$\mathrm{Log}(z_1 z_2) \neq \mathrm{Log} z_1 + \mathrm{Log} z_2$$

3. (15 pts) If ζ is a pole of a rational function R, then the coefficient of $1/(z-\zeta)$ in its partial fraction is called the residue of R at ζ and is denoted by $\mathrm{Res}(\zeta)$. Find $\mathrm{Res}(i)$ for $R(z) = \dfrac{2}{z^2+1}$.

4. (15 pts) Verify first that $u = xy - x + y$ is harmonic and then find a harmonic conjugate of u. And finally find an analytic function f whose real part is u.

5. (15 pts) Find $\displaystyle\int_\Gamma \frac{1}{z} dz$ for any contours in the right half-plane from $z = -3i$ to $z = 3i$. Justify your answer.

6. (15 pts, 10pts for Graduate Student) Use definition to prove that $\displaystyle\lim_{z\to -i} \frac{1}{z} = i$.

7. (15 pts, 10pts for Graduate Student) Use definition to prove the product rule of derivative: $(fg)' = f'g + fg'$.

8. (10 pts, **For Graduate Student Only**.) Let f be an analytic function with a continuous derivative satisfying $|f'(z)| \leq M$ for all z in the disk $D : |z| < 1$. Show that
$$|f(z_2) - f(z_1)| \leq M|z_2 - z_1|, \text{ for any } z_1, z_2 \text{ in } D.$$

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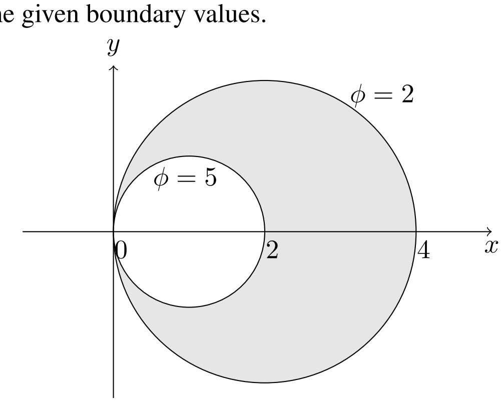
5 pts Bonus Question: Find an harmonic function ϕ between the circles, having the given boundary values.



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Solution Key

1. $1 - i\sqrt{3} = 2(1/2 - i\sqrt{3}/2) = 2(\cos(-\pi/3) + i\sin(-\pi/3)) = 2e^{i(-\pi/3+2k\pi)}$. Hence, $(1-i\sqrt{3})^{1/2} = \sqrt{2}e^{i(-\pi/6+k\pi)} = \pm(\sqrt{6}/2 - i\sqrt{2}/2)$ for $k = 0, 1$ respectively.

2. $\mathrm{Arg}i = \pi/2, \mathrm{Arg}(i-1) = 3\pi/4$, and $\mathrm{Arg}(-1-i) = -3\pi/4$, not $5\pi/4$ because the branch cut for $\mathrm{Arg}z$ is along $\tau = -\pi$ forcing $\mathrm{Arg}(z) \in (-\pi, \pi]$. Then $\mathrm{Arg}(z_1 z_2) = -3\pi/4 \neq \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) = \pi/2 + 3\pi/4$. Hence, $\mathrm{Log}(z_1 z_2) \neq \mathrm{Log}z_1 + \mathrm{Log}z_2$.

3. Either find the partial fraction of $2/(z^2+1) = A/(z-i) + B/(z+i)$ algebraically to get $A = -i, B = i$, and then by definition $\mathrm{Res}(i) = A = -i$, or by evaluation: $\lim_{z\to i}(z-i)R(z) = \lim_{z\to i}(A + B(z-i)/(z+i)) = A$ on the right and $\lim_{z\to i}(z-i)R(z) = \lim_{z\to i} 2/(z+i) = -i$ on the left. Either ways, $\mathrm{Res}(i) = A = -i$.

4. $u_x = y - 1, u_y = x + 1$ and $u_{xx} + u_{yy} = 0$. For the harmonic conjugate, by the C-R equations: $v_x = -u_y = -x - 1$ and $v_y = u_x = y - 1$, each implies $v = -x^2/2 - x + g(y)$ and $v = y^2/2 - y + h(x)$. When combined the information implies $v = -x^2/2 - 1 + y^2/2 - y + C$. And finally, $f(z) = u + iv = xy - x + y + i(-x^2/2 + y^2/2 - x - y)$.

5. $f(z) = 1/z$ has an antiderivative in the right half plane, $F(z) = \mathrm{Log}(z)$ or any $\log_\tau(z)$ for a branch cut τ in the left half plane. The by the Fundamental Theorem of Line Integral, $\int_\Gamma \frac{1}{z}dz = \mathrm{Log}(3i) - \mathrm{Log}(-3i) = \log(3) + i\pi/2 - (\log(3) - i\pi/2) = i\pi$.

6. First for any $\epsilon < 1$, and $\delta = \epsilon/2$, $|z + i| < \delta = \epsilon/2 < 1/2$ which implies $||z| - 1| = ||z| - |-i|| \leq |z - (-i)| < 1/2. \Rightarrow |z| - 1 > -1/2 \Rightarrow |z| > 1/2$. Now for $|z - (-i)| = |z + i| < \delta$ we have

$$\begin{aligned} |\frac{1}{z} - i| &= |\frac{1 - iz}{z}| = |\frac{(-i)(i + z}{z}| \\ &= \frac{|z + i|}{|z|} < \frac{|z + i|}{1/2} = 2|z + i| \\ &< 2\delta = \epsilon. \end{aligned}$$

By definition, this proves the limit.

7. (Identical to a write up from the solution key page to homework.) We will use the fact that a function is differentiable at a point is also continuous at the point. By definition, and a standard trick of adding and subtracting a same term, $f(z)g(z + \Delta z)$, in the numerator below, we have

$$\begin{aligned} [f(z)g(z)]' &= \lim_{\Delta z \to 0} \frac{f(z + \Delta z)g(z + \Delta z) - f(z)g(z)}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{f(z + \Delta z)g(z + \Delta z) - f(z)g(z + \Delta z) + f(z)g(z + \Delta z) - f(z)g(z)}{\Delta z} \\ &= \lim_{\Delta z \to 0} \left[\frac{f(z + \Delta z) - f(z)}{\Delta z} g(z + \Delta z) + f(z) \frac{g(z + \Delta z) - g(z)}{\Delta z} \right] \\ &= \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} \lim_{\Delta z \to 0} g(z + \Delta z) + f(z) \lim_{\Delta z \to 0} \frac{g(z + \Delta z) - g(z)}{\Delta z} \\ &= f'(z)g(z) + f(z)g'(z). \end{aligned}$$

The last equality follows from the assumption and the fact that all the limits involved exist, as well as the production and summation rules of limit. This proves the result.

8. By the Fundamental Theorem of Calculus, $f(z_2) - f(z_1) = \displaystyle\int_{z_1}^{z_2} f'(z)dz$, where the path is the straight line from z_1 to z_2 lying entirely in D. The path length

is $L = |z_2 - z_1|$. Since $|f'(z)| \leq M$ in D, we have from the theorem on integral estimate:

$$f(z_2) - f(z_1) = |\int_{z_1}^{z_2} f'(z)dz| \leq \int_{z_1}^{z_2} |f'(z)||dz|$$
$$\leq M \int_{z_1}^{z_2} |dz| = ML = M|z_2 - z_1|.$$

This proves the result.

Bonus question. For $1/z = u(x,y) + iv(x,y)$, u, v are harmonic in the plane minus the origin. Also the level curves of $u(x,y) = x/(x^2 + y^2)$ are circles centered on the x-axis and through the origin. Thus, the solution to the Laplace equation in the domain with the boundary conditions are $\phi(x,y) = Au(x,y) + B$. Since the inner circle is $u(x,y) = 1/2$ and the outer circle is $u(x,y) = 1/4$, we have $A = 6, B = 1/2$.