Math 423/823 Spring 2014 Final Exam

Circle One: Math423 **or** Math823 **Name:** ____________________ **Score:** ________

Instructions: You must show supporting work to receive full and partial credits. No text book, notes, or formula sheets allowed. If you registered for Math423, do #1-12. If you registered for Math823, do #3-14.

1. (20 pts, **For Math423 Student Only**.) Let C be the circle $|z| = 2$ traversed once in the positive direction. Find

$$\int_C \frac{e^{-z}}{(z+1)^2} dz$$

2. (20 pts, **For Math423 Student Only**.) Find all four roots of the equation $z^4 + 1 = 0$ and use them to deuce the factorization $z^4 + 1 = (z^2 - \sqrt{2}z + 1)(z^2 + \sqrt{2}z + 1)$.

3. (20 pts) Show that $h(z) = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y)$ is differentiable on the coordinate axes but nowhere analytic.

4. (20 pts) If $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ is a polynomial and $\max|p(z)| = M$ for $|z| = 1$, show that the coefficient a_0 is bounded by M.

5. (15 pts) Find the Taylor series for $\dfrac{1+z}{1-z}$ around $z_0 = i$ and state the radius of convergence.

6. (15 pts) Find the Laurent series for the function $\dfrac{1}{z}$ in the domain $1 < |z+1|$.

7. (15 pts) Prove that all the roots of the equation $z^6 - 5z^2 + 10 = 0$ lie in the annulus $1 < |z| < 2$.

8. (15 pts) Use the method of residues to verify this integral $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2} = \pi$.

9. (15 pts) Use the method of residues to verify this integral $\displaystyle\int_0^{2\pi} \frac{d\theta}{2 - \cos\theta} = \frac{2\pi}{\sqrt{3}}$.

10. (15 pts) The full version of the Rouché's Theorem is: If f is analytic and nonzero at each point of a simple closed positively oriented contour C and at any interior point f is either analytic or f has a pole, then

$$\frac{1}{2\pi i} \oint_{|z|=3} \frac{f'(z)}{f(z)} dz = N_0(f) - N_p(f),$$

where $N_0(f)$ and $N_p(f)$ are, respectively, the number of zeros and poles of f inside C, counting multiplicity. Use this result as given to evaluate

$$\frac{1}{2\pi i} \oint_{|z|=3} \frac{f'(z)}{f(z)} dz, \text{ where } f(z) = \frac{z^2(z-i)^3 e^z}{3(z+2)^4(3z-18)^5}.$$

11. (15 pts) What is the image of the region $1 < \text{Im} z$ under the mapping $w = \dfrac{z-i}{z}$? (*Hint*: Either do it in steps by a sequence of simpler mappings or do it by establishing the boundary correspondence and some special points.)

12. (15 pts) Suppose that f is analytic inside and on the simple closed curve C and that $|f(z) - 1| < 1$ for all z on C. Prove that f has no zeros inside C. (Hint: it can be proved either by the maximum principle or by Rouché's Theorem.)

13. (20 pts, **For Math823 Student Only**.) Prove the Fundamental Theory of Algebra: Every polynomial of degree n has n roots, including multiplicity.

14. (20 pts, **For Math823 Student Only**.) Show that in polar coordinates (r, θ) Laplace's equation becomes

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

2 Bonus Points: True or False: The Nebraska State Book is the Bible. (... The End)