Sec.1.2.

14. Prove that $|z_1 z_2| = |z_1||z_2|$.

Proof: Since $z^2 = z\overline{z}$ and $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$, then by the commutativity and associativity of multiplication we have $|z_1 z_2|^2 = (z_1 z_2)(\overline{z_1 z_2}) = (z_1\overline{z_1})(z_2\overline{z_2}) = |z_1|^2|z_2|^2$, which implies the identity.

Sec.1.3.

16. Prove that $||z_1| - |z_2|| \leq |z_1 - z_2|$.

Proof: Let $z = z_1 - z_2$ and $w = z_2$. Since $z + w = z_1$, we have by the triangle inequality that $|z_1| = |z+w| \leq |z|+|w| = |z_1-z_2|+|z_2|$. Hence, $|z_1|-|z_2| \leq |z_1-z_2|$. Exchanging the roles of z_1 and z_2, we have $-(|z_1| - |z_2|) = |z_2| - |z_1| \leq |z_2 - z_1| = |-(z_1 - z_2)| = |z_1 - z_2|$, equivalently, $-|z_1 - z_2| \leq |z_1| - |z_2|$. Together, the inequalities $-|z_1 - z_2| \leq |z_1| - |z_2|$ and $|z_1| - |z_2| \leq |z_1 - z_2|$ is equivalent to $||z_1| - |z_2|| \leq |z_1 - z_2|$. The end of proof.

Sec.1.6.

10. Prove that the closed disk $|z - z_0| \leq \rho$ is bounded.

Proof: For every z satisfying $|z - z_0| \leq \rho$ we have $||z| - |z_0|| \leq |z - z_0| \leq \rho$ which is $-\rho \leq |z| - |z_0| \leq \rho$. The second inequality becomes $|z| \leq |z_0| + \rho := N$, showing that z is bounded by N.

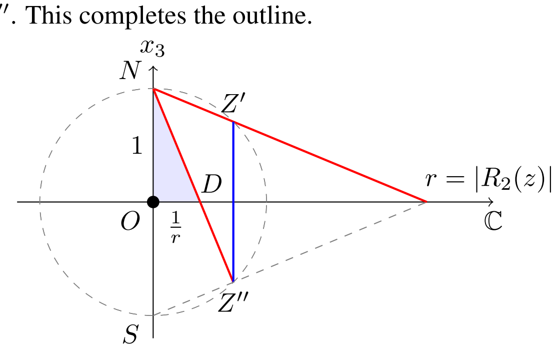
13. Let S be a subset of $\mathbb{C}$. Prove that S is closed if and only if its complement $S^c := \mathbb{C} \setminus S$ is open.

Proof: Denote by ∂D the boundary of a set D. It is straightforward to show by definition that $\partial S = \partial S^c$ because $\forall z \in \partial S$ and $\forall \epsilon > 0,\ D_\epsilon(z) \cap S \neq \emptyset$ and $D_\epsilon(z) \cap S^c \neq \emptyset$ which by definition is equivalent to $z \in S^c$. To show S^c is closed if S is open we only need to show $\forall z \in \partial S^c \Rightarrow z \in S^c$. This is true because if $z \in S$ then $\exists D_\rho(z) \subset S$ for some $\rho > 0$ since S is open, an contradiction. To show S is open if S^c is closed we only need to show for each $z \in S$ there is a ρ so that $D_\rho(z) \subset S$. This is true because $z \notin S^c \Rightarrow z \notin \partial S^c = \partial S \Rightarrow z \notin \partial S \Rightarrow$ there must be one $\rho > 0$ so that $D_\rho(z) \subset S$. Otherwise, we would have for all small $\epsilon > 0$ there must be at least one point in $D_\epsilon(z) \cap S^c$, a contradiction.

Sec.1.7.

8. Second part: give a geometric explanation of $\chi(z, w) = \chi(1/z, 1/w)$.

Outline of Explanation: Let R_2 be the reflection operation with respect to the x_1x_3-plane in $\mathbb{R}^3$, $R_2(Z) = R_2(x_1, x_2, x_3) = (x_1, -x_2, x_3)$ and similarly R_3 be the reflection operation with respect to the x_1x_2-plane with $R_3(Z) = R_3(x_1, x_2, x_3) = (x_1, x_2, -x_3)$. Let $Z = P(z)$ denote the stereographic projection. We note first that both R_2 and R_3 preserve the Euclidean distance: $d(Z, W) = d(R_i(Z), R_i(W))$. We note second that the x_1x_3-reflection reserves the stereographic projection, i.e. $R_2(P(z)) = P(R_2(z))$, because R_2 transforms $z, Z = P(z)$ and its defining line through the north pole to the corresponding reflections. We should also note that the two reflection operations map points from the unit sphere to points on the unit sphere. Let $Z = P(z), Z' = R_2(Z), Z'' = R_3(Z')$ and similar notations for w. We know $d(Z,W) = d(Z',W') = d(Z'',W'')$. It only remains to show this claim that $P(\frac{1}{z}) = Z'' = R_3(R_2(P(z)))$, which implies $\chi(z,w) = d(Z,W) = d(Z'',W'') = \chi(1/z, 1/w)$. Use the polar coordinate form $z = re^{i\theta}$ to get $1/z = \frac{1}{r}e^{-\theta}$, a point on the radial plane with the phase angle $-\theta$ in $\mathbb{R}^3$. The reflected point $z' = R_2(z)$ and $Z' = R_2(Z)$ lie on the same radial plane as well. The claim is proved by using Euclidean geometry as depicted in the diagram below. It is a cross-section view on the radial plan $\text{Arg}z = -\theta$ for the case of $r \geq 1$. From which we see the following: $\Delta NOD \sim \Delta NSZ''$ because they are right triangles and share a common angle. Also $\Delta NSZ'' \sim \Delta NOr$ because they are right triangles and because the angle $\angle NSZ'' = \angle SNr$ as they are from the isosceles triangle ΔNSr. Therefore $\Delta NOD \sim \Delta NOr$ which implies $D/1 = 1/r$ by similarity. Thus $P(1/z) = Z''$. For the case of $r < 1$, the same diagram works after exchanging r with $1/r$ and Z' with Z''. This completes the outline.



Sec.2.2.

Theorem 1(ii): If $\lim_{z\to z_0} f(z) = A,\ \lim_{z\to z_0} g(z) = B$, then $\lim_{z\to z_0} f(z)g(z) = AB$.

Proof: We first show that If $\lim_{z\to z_0} f(z) = A$ then f is bounded by a constant M in a disk $D_\rho(z_0)$ for some $\rho > 0$. In fact, by definition for $\epsilon = 1$ there is a $\rho > 0$ so that $|z-z_0| < \rho \Rightarrow |f(z)-A| < \epsilon = 1$ which implies $||f(z)|-|A|| \leq |f(z)-A| < 1$ and $|f(z)| < |A|+1 := M$ follows. Thus, without loss of generality (wlog), we can assume $|f(z)|, |g(z)| \leq N$ for $z \in D_r(z_0)$ for some number $N > \max |A|+1, |B|+1$ and $r > 0$.

We first use the assumption to have $\forall \epsilon > 0 \ \exists \ \delta_1 > 0$ and $\delta_2 > 0$ with $\delta_{1,2} < r$ s.t.(such that) $|z - z_0| < \delta_1 \Rightarrow |f(z) - A| < \epsilon/(2N)$ and $|z - z_0| < \delta_2 \Rightarrow |g(z) - B| < \epsilon/(2N)$. Now let $\delta = \min\{\delta_1, \delta_2\}$ then $\forall |z - z_0| < \delta \leq \delta_1, \delta_2$ both $|f(z) - A| < \epsilon/(2N)$ and $|g(z) - B| < \epsilon/(2N)$.

Now for the product rule, we have $\forall z$ with $|z - z_0| < \delta$

$$|f(z)g(z)-AB| = |f(z)g(z)-Ag(z)+Ag(z)-AB| \leq |f(z)-A||g(z)|+|A||g(z)-B|$$

$$\leq N(|f(z) - A| + |g(z) - B|) < N[\epsilon/(2N) + \epsilon/(2N)] = \epsilon.$$

This proves the result.

22. Show that if $\lim_{n\to\infty} f(z_n) = w_0$ for every sequence $\{z_n\}_1^\infty$ converging to z_0 ($z_n \neq z_0$), then $\lim_{z\to z_0} f(x) = w_0$.
Proof: Suppose on the contrary that $\lim_{z\to z_0} f(x) \neq w_0$. Then by definition there must be a number $\epsilon_0 >$ so that for every $\delta > 0$ the inequality $|f(z) - w_0| < \epsilon_0$ does not hold for every point $|z - z_0| < \delta$. That is there is some point z_δ with $|z_\delta - z_0| < \delta$ but $|f(z_\delta) - w_0| \geq \epsilon_0$. In particular, for each integer n we can find a point z_n satisfying $|z_n - z_0| < 1/n$ but $|f(z_n) - w_0| \geq \epsilon_0$. This contradicts the assumption because by construction this sequence $z_n \to z_0$ but $\lim_{n\to\infty} f(z_n) \neq w_0$.

Sec.2.4.

Prove the product rule of derivative: $[f(z)g(z)]' = f'(z)g(z) + f(z)g'(z)$
Proof: We will use the fact that a function is differentiable at a point is also continuous at the point. By definition, and a standard trick of adding and subtracting a same term, $f(z)g(z + \Delta z)$, in the numerator below, we have

$$\begin{aligned}
[f(z)g(z)]' &= \lim_{\Delta z\to 0} \frac{f(z+\Delta z)g(z+\Delta z) - f(z)g(z)}{\Delta z} \\
&= \lim_{\Delta z\to 0} \frac{f(z+\Delta z)g(z+\Delta z) - f(z)g(z+\Delta z) + f(z)g(z+\Delta z) - f(z)g(z)}{\Delta z} \\
&= \lim_{\Delta z\to 0} \left[\frac{f(z+\Delta z) - f(z)}{\Delta z} g(z+\Delta z) + f(z)\frac{g(z+\Delta z) - g(z)}{\Delta z}\right] \\
&= \lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} \lim_{\Delta z\to 0} g(z+\Delta z) + f(z) \lim_{\Delta z\to 0} \frac{g(z+\Delta z) - g(z)}{\Delta z} \\
&= f'(z)g(z) + f(z)g'(z).
\end{aligned}$$

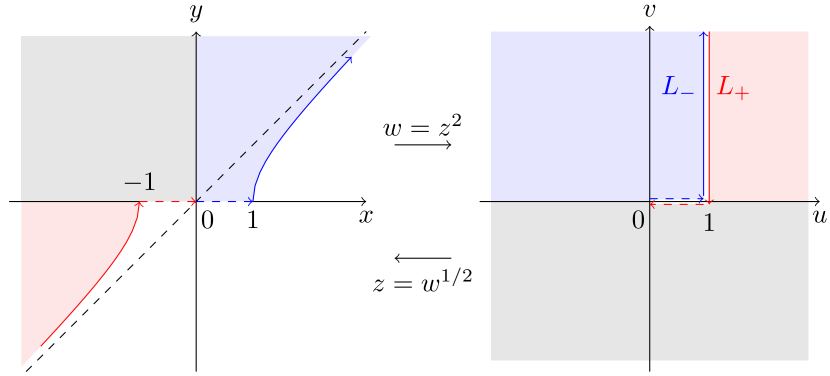
The last equality follows from the assumption and the fact that all the limits involved exist, as well as the production and summation rules of limit. This proves the result.

11 Suppose that $f(z)$ and $\overline{f(z)}$ are analytic in a domain. Show that $f(z)$ is constant in D.
Proof: Let $f(z) = u(x, y) + iv(x, y)$ and $g(z) = f(z) + \overline{f(z)} = 2u(x, y)$. Then by the summation rule, g is also analytic in D. Since $\text{Im}g(z) = 0$, by the C-R equations we have $u_x = u_y = 0$ and hence $u(x, y)$ is constant, which in turn implies $v(x, y)$ is constant, and hence the proof.

Sec.3.3.

Lecture example: Find the domain of the function $w = z^2$ whose range is the plane with the branch cut $L = \{\text{Re}w = 1, \text{Im}w \in [0, +\infty)\}$.
Solution: The branch cut L has two sides. On the left side $w \in L_-$, we have $\arg w = \text{Arg}w \in [0, \pi/2)$. On the right side $w \in L_+$, we have $\arg w = \text{Arg}w + 2\pi \in [2\pi, 2\pi + \pi/2)$. To find the preimage of $L_\pm$ is to find the image of the inverse map $z = w^{1/2} = |w|^{1/2}e^{i(\arg w)/2} = |w|^{1/2}e^{i(\text{Arg}w/2+k\pi)}$. Let $w = u + iv$ and $\text{Arg}w = \theta$. Then for $w \in L_-$ with $k = 0$, $|w| = \sqrt{1+v^2}$, and $\cos(\theta/2) = \sqrt{\frac{1+\cos(\theta)}{2}}$, $\sin(\theta/2) = \sqrt{\frac{1-\cos(\theta)}{2}}$ by the half-angle formula. As $\tan\theta = v/u = v$, we have $\cos(\theta) = 1/\sqrt{1+v^2}$ (and $\sin(\theta) = v/\sqrt{1+v^2}$). Therefore $x + iy = z = w^{1/2} =$

$(1+v^2)^{1/4}[\cos(\theta/2)+i\sin(\theta/2)] = (1+v^2)^{1/4}[\sqrt{\frac{\sqrt{1+v^2}+1}{2\sqrt{1+v^2}}}+i\sqrt{\frac{\sqrt{1+v^2}-1}{2\sqrt{1+v^2}}}]$, which is simplified to $x = s(v) = \sqrt{\frac{\sqrt{1+v^2}+1}{2}}$ and $y = t(v) = \sqrt{\frac{\sqrt{1+v^2}-1}{2}}$ parameterized by parameter $v \in [0,\infty)$. This obtains L_- corresponding side of a cut in the domain D. For the L_+ side of the cut in the range, since $\arg w = \mathrm{Arg} w + 2\pi$, the image under $w^{1/2}$ differs from that of L_- exactly by a negative sign, that is $x = -s(v), y = -t(v)$. See figure, in which the additional cut $\{0 \le u \le 1,\ v = 0\}$ is needed for the definition of the inverse function $z = w^{1/2}$. Also the corresponding subregions in domain and range are color coded.



Sec.3.3.

10. Show that the function $\mathrm{Log}(-z) + i\pi$ is a branch of $\log z$ in the domain D_0 consisting of all points in the plane except those on the nonnegative real axis.

Solution: By definition, $\mathrm{Log}(-z) + i\pi = \log|z| + i(\mathrm{Arg}(-z) + \pi)$. There are two cases as $\mathrm{Arg}(z) \in (-\pi, \pi]$. For z in the quadrant I or II, $-z$ is in quadrant III or IV, $\mathrm{Arg}(-z) = \mathrm{Arg}(z) - \pi$. For z in the quadrant III or IV, $-z$ is in quadrant I or II, $\mathrm{Arg}(-z) = \mathrm{Arg}(z) + \pi$. Therefore, $\mathrm{Arg}(-z) + \pi = \mathrm{Arg}(z) = \arg_0(z) \in (0, \pi)$ if z is in quadrant I or II, and $\mathrm{Arg}(-z) + \pi = \mathrm{Arg}(z) + 2\pi = \arg_0(z) \in (\pi, 2\pi)$ if z in the quadrant III or IV. Together, this shows $\mathrm{Log}(-z) + i\pi = \log_0(z)$.